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Analysis of XY Model with Mexican-Hat Interaction on a Circle – Derivation of Saddle Point Equations and Study of Bifurcation Structure –

Tatsuya Uezu*, Tomoyuki Kimoto[1], and Masato Okada[2,3]

Graduate School of Sciences and Humanities, Nara Women's University, Nara 630, Japan
[1]*Oita National College of Technology, Oita 870-0152, Japan*
[2]*Graduate School of Frontier Sciences, The University of Tokyo, Kashiwa, Chiba 277-8561, Japan*
[3]*RIKEN Brain Science Institute, Wako, Saitama 351-0198, Japan*

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In our previous study, we investigated a classical XY model on a circle by adopting the Mexican-hat type interaction, which is composed of uniform and location-dependent interactions. We solved the saddle point equations numerically and found three nontrivial solutions. In this study, we determined the phases of complex order parameters and derived the saddle point equations for stable and unstable nontrivial solutions and the formula of boundaries of bistable regions analytically. We performed Markov Chain Monte Carlo simulations and confirmed that the numerical and theoretical results agree well.

KEYWORDS: XY model, Mexican-hat interaction, saddle point equations, bistability

§1. Introduction

Over these past years, we have been studying the synchronization - desynchronization phase transition of oscillator networks.[1] In particular, we have studied the phase oscillator network[2–4] with the Mexican-hat type interaction on a circle. This type of interaction was introduced to model the feature extraction cells in neurosciences[5,6] and to express effects of excitation of nearby neurons and inhibition of distant neurons.

In the course of the analysis of the phase oscillator network, it turned out that information on the phases of complex order parameters is necessary. Therefore, we studied the XY model on a circle with the same interaction as the phase oscillator network, because both models coincide with each other under some conditions.

In the XY model, we found three nontrivial solutions of the saddle point equations (SPEs), the uniform (U), spinning (S), and pendulum (Pn) solutions.[7] We confirmed the agreement between the theoretical and numerical results, and drew phase diagrams by performing numerical simulations.

In this study, we theoretically determined the phases of complex order parameters that enabled us to derive the self-consistent equations (SCEs) of the amplitudes of complex order parameters in the phase oscillator network. We derived the SPEs of the amplitudes of complex order param-

* E-mail address: uezu@ki-rin.phys.nara-wu.ac.jp

eters for stable and unstable nontrivial solutions as well. Furthermore, we derived the formula of boundaries of bistable regions by identifying and using the unstable Pn solution. We performed Markov Chain Monte Carlo (MCMC) simulations and found that the numerical and theoretical results agree well.

The structure of this paper is as follows. In §2, we formulate the model. In §3 and §4, we analyze the model with only the location-dependent interaction, and the model with both the uniform and location-dependent interactions, respectively. We derive the formula of boundaries of bistable regions in §5. Summary and discussion are given in §6. In Appendix, we give the detailed derivation of the SPEs.

§2. Formulation

Let us consider the classical XY model. We assume that the magnitude of the XY spin $\boldsymbol{X} = (X, Y)$ is 1. Let ϕ_i be the phase of the i-th spin $\boldsymbol{X}_i = (X_i, Y_i)$,

$$X_i = \cos\phi_i, \quad Y_i = \sin\phi_i. \tag{1}$$

The Hamiltonian H and the interaction J_{ij} between the i-th and j-th spins are given by

$$H = -\sum_{i<j} J_{ij}\cos(\phi_i - \phi_j), \tag{2}$$

$$J_{ij} = \frac{J_0}{N} + \frac{J_1}{N}\cos(\theta_i - \theta_j), \quad \theta_i = i\frac{2\pi}{N}, i = 0, \cdots, N-1. \tag{3}$$

Here, θ_i is the coordinate of the i-th spin on the unit circle. The interaction J_{ij} has the property of the Mexican-hat type interaction. Now, we introduce the following three complex order parameters:

$$W = Re^{i\Theta} = \frac{1}{N}\sum_j e^{i\phi_j}, \tag{4}$$

$$W_c = R_c e^{i\Theta_c} = \frac{1}{N}\sum_j \cos\theta_j e^{i\phi_j}, \tag{5}$$

$$W_s = R_s e^{i\Theta_s} = \frac{1}{N}\sum_j \sin\theta_j e^{i\phi_j}. \tag{6}$$

By using R and $R_1 = \sqrt{R_c^2 + R_s^2}$, H is rewritten as

$$H = -\frac{N}{2}(J_0 R^2 + J_1 R_1^2) + \frac{1}{2}(J_0 + J_1). \tag{7}$$

We introduce different expressions of order parameters as

$$R_R = R\cos\Theta = \frac{1}{N}\sum_j \cos\phi_j, \ R_I = R\sin\Theta = \frac{1}{N}\sum_j \sin\phi_j,$$

$$R_{cR} = R_c\cos\Theta_c = \frac{1}{N}\sum_j \cos\theta_j\cos\phi_j, \ R_{cI} = R_c\sin\Theta_c = \frac{1}{N}\sum_j \cos\theta_j\sin\phi_j,$$

$$R_{sR} = R_s\cos\Theta_s = \frac{1}{N}\sum_j \sin\theta_j\cos\phi_j, \ R_{sI} = R_s\sin\Theta_s = \frac{1}{N}\sum_j \sin\theta_j\sin\phi_j.$$

Introducing their conjugate variables and using the relations

$$\int dR_R \delta(R_R - \frac{1}{N}\sum_j \cos\phi_j) = 1,$$
$$\delta(R_R - \frac{1}{N}\sum_j \cos\phi_j) = \frac{N}{2\pi i}\int d\hat{R}_R e^{-N\hat{R}_R(R_R - \frac{1}{N}\sum_j \cos\phi_j)},$$

the partition function Z is expressed as

$$\begin{aligned}
Z &= \mathrm{Tr}\exp[-\beta H] = \mathrm{Tr}\exp[\beta\frac{N}{2}(J_0R^2 + J_1R_1^2) - \frac{\beta}{2}(J_0 + J_1)] \\
&= e^{-\frac{\beta}{2}(J_0+J_1)}\left(\frac{N}{2\pi i}\right)^6 \int d\boldsymbol{R} e^{NG}, \\
G &= G_0 + G_1, \\
G_0 &= \frac{\beta}{2}(J_0R^2 + J_1R_1^2) \\
&\quad -(\hat{R}_R R_R + \hat{R}_I R_I + \hat{R}_{cR}R_{cR} + \hat{R}_{cI}R_{cI} + \hat{R}_{sR}R_{sR} + \hat{R}_{sI}R_{sI}), \\
e^{NG_1} &= \exp[\sum_j \ln\int d\phi_j \exp\{A_j\cos\phi_j + B_j\cos\phi_j\}], \\
A_j &= \hat{R}_R + \hat{R}_{cR}\cos\theta_j + \hat{R}_{sR}\sin\theta_j, \\
B_j &= \hat{R}_I + \hat{R}_{cI}\cos\theta_j + \hat{R}_{sI}\sin\theta_j, \\
&\mathrm{Tr} = \int d\boldsymbol{\phi} = d\phi_1 d\phi_2\cdots d\phi_N, \\
&d\boldsymbol{R} = d\hat{R}_R dR_R d\hat{R}_I dR_I d\hat{R}_{cR} dR_{cR} d\hat{R}_{cI} dR_{cI} d\hat{R}_{sR} dR_{sR} d\hat{R}_{sI} dR_{sI}.
\end{aligned}$$

Here, we put $\beta = \frac{1}{T}$, and T is 'temperature'. Under optimal conditions of G with respect to R_R, R_I, and so forth, we obtain

$$\begin{aligned}
&\hat{R}_R = \beta J_0 R_R,\ \hat{R}_I = \beta J_0 R_I, \\
&\hat{R}_{cR} = J_1 R_{cR},\ \hat{R}_{cI} = J_1 R_{cI}, \hat{R}_{sR} = J_1 R_{sR},\ \hat{R}_{sI} = J_1 R_{sI}.
\end{aligned} \tag{8}$$

Thus, G_0 is expressed as

$$G_0 = -\frac{\beta}{2}(J_0R^2 + J_1R_1^2). \tag{9}$$

By introducing C_j and ϕ_j^0 as

$$\begin{aligned}
&A_j\cos\phi_j + B_j\sin\phi_j = C_j\cos(\phi_j - \phi_j^0), \\
&C_j = \sqrt{A_j^2 + B_j^2},\ C_j\cos\phi_j^0 = A_j, C_j\sin\phi_j^0 = B_j,
\end{aligned}$$

G_1 is now expressed by

$$\begin{aligned}
G_1 &= \frac{1}{N}\sum_j \ln\int d\phi_j e^{A_j\cos\phi_j + B_j\sin\phi_j} = \frac{1}{N}\sum_j \ln\int d\phi_j e^{C_j\cos(\phi_j - \phi_j^0)} \\
&= \frac{1}{N}\sum_j \ln\{2\pi I_0(\beta\Xi(\theta))\} = \frac{1}{2\pi}\int_0^{2\pi} d\theta \ln\{2\pi I_0(\beta\Xi(\theta))\},
\end{aligned} \tag{10}$$

where $I_n(z)$ and $\Xi(\theta_j)$ are defined by

$$I_n(z) = \frac{1}{2\pi}\int_0^{2\pi} d\phi \cos n\phi \; e^{z\cos\phi}, \tag{11}$$

$$\begin{aligned}\Xi(\theta_j) &= \sqrt{(\frac{A_j}{\beta})^2+(\frac{B_j}{\beta})^2} \\ &= [\{J_0R_R + J_1\left(R_{cR}\cos\theta_j + R_{sR}\sin\theta_j\right)\}^2 \\ &\quad +\{J_0R_I + J_1\left(R_{cI}\cos\theta_j + R_{sI}\sin\theta_j\right)\}^2]^{1/2}.\end{aligned} \tag{12}$$

By introducing $\tilde{\Theta}_c \equiv \Theta_c - \Theta$ and $\tilde{\Theta}_s \equiv \Theta_s - \Theta$, $\Xi(\theta)$ is further rewritten as

$$\begin{aligned}\Xi(\theta)^2 &= \\ (J_0R)^2 + J_1^2\{(R_c\cos\theta)^2 &+ (R_s\sin\theta)^2 + 2R_cR_s\cos(\tilde{\Theta}_c-\tilde{\Theta}_s)\sin\theta\cos\theta\} \\ &+2J_0J_1R\{R_c\cos\tilde{\Theta}_c\cos\theta + R_s\cos\tilde{\Theta}_s\sin\theta\}.\end{aligned} \tag{13}$$

The free energy f per spin is expressed by

$$f = -\frac{1}{\beta N}\ln Z = -\frac{1}{\beta N}G. \tag{14}$$

Therefore, G and f depend only on $R, R_c, R_s, \tilde{\Theta}_c$, and $\tilde{\Theta}_s$.

2.1 $J_0 > 0, J_1 = 0$, the case of ferromagnetic interactions

In this case, $\Xi(\theta)$ and f are expressed by

$$\Xi(\theta) = J_0R \tag{15}$$

$$f = \frac{1}{2}J_0R^2 - \frac{1}{\beta}\ln\{2\pi I_0\left(\beta J_0 R\right)\}, \tag{16}$$

and SPE becomes

$$R = \frac{I_1(\beta J_0R)}{I_0(\beta J_0R)}. \tag{17}$$

It turns out that this is the stable U solution in which $R > 0$ and $R_1 = 0$. The critical temperature is given by

$$T_{0,c} = \frac{J_0}{2}. \tag{18}$$

§3. Case of $J_0 = 0$ and $J_1 > 0$

$\Xi(\theta)$ and f are given by

$$\Xi(\theta) = J_1\sqrt{(R_c\cos\theta)^2+(R_s\sin\theta)^2+2R_cR_s\cos\hat{\Theta}\sin\theta\cos\theta}, \tag{19}$$

$$f = \frac{1}{2}J_1R_1^2 - \frac{1}{\beta}\frac{1}{2\pi}\int_0^{2\pi} d\theta\ln\{2\pi I_0\left(\beta\Xi(\theta)\right)\}, \tag{20}$$

where $\hat{\Theta} = \tilde{\Theta}_c - \tilde{\Theta}_s$. f depends only on R_c, R_s, and $\hat{\Theta}$. From the optimal condition of f with respect to $\hat{\Theta}$, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{I_1(\beta\Xi)}{I_0(\beta\Xi)}\frac{1}{\Xi}\sin\theta\cos\theta\sin\hat{\Theta} = 0. \tag{21}$$

The following two cases are deduced under this condition:

$$\textbf{Case 1} \qquad \sin\hat{\Theta} = 0, \tag{22}$$

$$\textbf{Case 2} \qquad \frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{I_1(\beta\Xi)}{I_0(\beta\Xi)}\frac{1}{\Xi}\sin\theta\cos\theta = 0. \tag{23}$$

In the following, we study these cases separately.

3.1 Case 1

From the condition $\sin\hat{\Theta} = 0$, $\hat{\Theta} = 0$ and π follow in mod 2π. Thus, we have the equation

$$R_1 = \frac{2}{\pi}\int_0^{\pi/2} d\theta \frac{I_1(\beta J_1 R_1\cos\theta)}{I_0(\beta J_1 R_1 \cos\theta)}\cos\theta. \tag{24}$$

It turns out that this is the equation for an unstable solution, because simulation results do not agree with the solution of this equation.

3.2 Case 2

We change the integration range from $[0, 2\pi]$ to $[-\pi, \pi]$ for convenience in eq. (23). The necessary and sufficient condition for eq. (23) is that the Fourier series expansion of the integrand does not contain the term $\sin(2\theta)$. That is, the condition is

$$R_c R_s \cos\hat{\Theta} = 0. \tag{25}$$

This implies

$$\hat{\Theta} = \pm\frac{\pi}{2} \text{ or } R_c = 0, \text{ or } R_s = 0. \tag{26}$$

The SPEs become

$$\Xi(\theta) = J_1\hat{\Xi}(\theta), \tag{27}$$

$$\hat{\Xi}(\theta) = \sqrt{R_c^2\cos^2\theta + R_s^2\sin^2\theta}, \tag{28}$$

$$R_c = R_c\frac{1}{\pi}\int_0^{\pi} d\theta \frac{I_1(\beta J_1\hat{\Xi})}{I_0(\beta J_1\hat{\Xi})}\frac{1}{\hat{\Xi}}\cos^2(\theta), \tag{29}$$

$$R_s = R_s\frac{1}{\pi}\int_0^{\pi} d\theta \frac{I_1(\beta J_1\hat{\Xi})}{I_0(\beta J_1\hat{\Xi})}\frac{1}{\hat{\Xi}}\sin^2(\theta). \tag{30}$$

Let us consider three cases separately.

Case of $\hat{\Theta} = \pm\frac{\pi}{2}$

Let us define θ_0 as

$$R_1\cos\theta_0 = R_c, \quad R_1\sin\theta_0 = R_s, \quad 0 \le \theta_0 \le \frac{\pi}{2}. \tag{31}$$

Then, defining $\bar{\Xi}(\theta) \equiv \sqrt{1+\cos(2\theta_0)\cos(2\theta)}$, we have

$$\hat{\Xi}(\theta) \quad = \quad \frac{R_1}{\sqrt{2}}\sqrt{1+\cos(2\theta_0)\cos(2\theta)} = \frac{R_1}{\sqrt{2}}\bar{\Xi}(\theta). \tag{32}$$

Under the optimal condition of f with respect to θ_0, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta\beta J_1 R_1 \frac{1}{\sqrt{2}\bar{\Xi}}(-\sin 2\theta_0 \cos 2\theta)\frac{I_1(\beta J_1 R_1 \bar{\Xi}/\sqrt{2})}{I_0(\beta J_1 R_1 \bar{\Xi}/\sqrt{2})} = 0. \tag{33}$$

Thus, the necessary and sufficient condition for this is $\sin 2\theta_0 = 0$ or the coefficient of $\cos 2\theta$ in $\bar{\Xi}$ is 0. Therefore,

$$\sin 2\theta_0 = 0 \text{ or} \cos 2\theta_0 = 0. \tag{34}$$

When $\sin 2\theta_0 = 0$, $\theta_0 = 0$ or $\frac{\pi}{2}$ follows. Then, $R_c = R_1, R_s = 0$, or $R_s = R_1, R_c = 0$ follows. When $\cos 2\theta_0 = 0$, $\theta_0 = \frac{\pi}{4}$ follows, and we obtain $R_c = R_s = \frac{1}{\sqrt{2}}R_1$.

Case of $R_c = 0$, or $R_s = 0$

This case already appears in the previous case.

Therefore, the possible solutions for case 2 are $(R_c = R_1, R_s = 0)$ or $(R_s = R_1, R_c = 0)$ or $(R_c = R_s = \frac{1}{\sqrt{2}}R_1)$.

For the case of $(R_c = R_1, R_s = 0)$ or $(R_s = R_1, R_c = 0)$, the SPE turns out to be the same as in case 1, eq. (24).

For the last case, $R_c = R_s = \frac{1}{\sqrt{2}}R_1$, the SPE is

$$R_c \quad = \quad \frac{1}{2}\frac{I_1(\beta J_1 R_c)}{I_0(\beta J_1 R_c)}. \tag{35}$$

The critical temperature is given by

$$T_{1,c} \quad = \quad \frac{J_1}{4}. \tag{36}$$

Numerical results. We performed MCMC simulations. In Fig. 1, we display the theoretical and simulation results for $J_0 = 0$ and $J_1 = 1$. The agreement between the theoretical result (eq. (35)) and simulation result is good. That is, eq. (35) is the SPE of the stable S solution in which $R = 0$ and $R_1 > 0$.

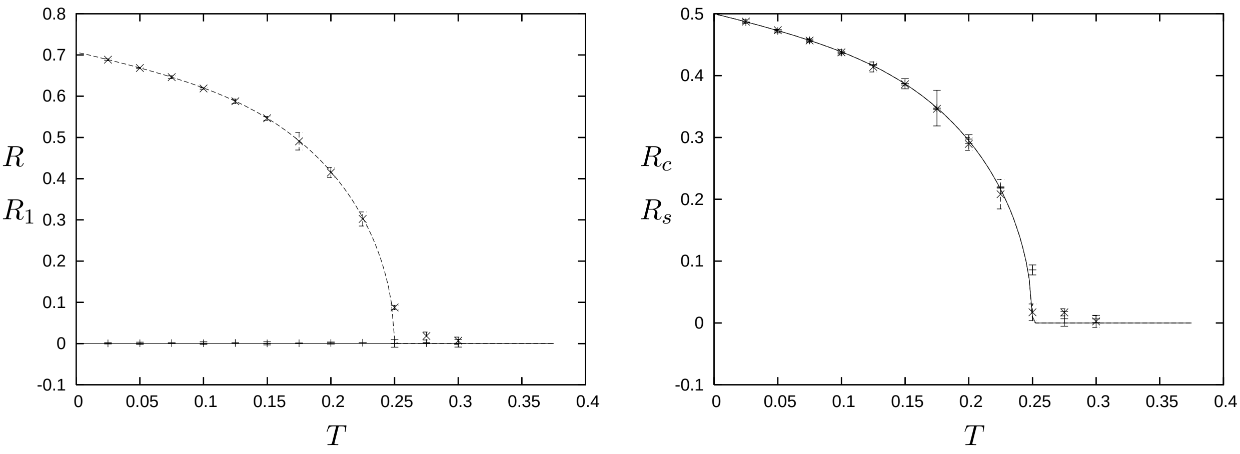

Fig. 1. Temperature dependences of order parameters. $N = 10000$. Curves: theory, eq. (35). Symbols: simulation. (a) Solid curve and $+$: R. Dashed curve and $\times$: R_1. (b) Solid curve denotes R_c and R_s. $+$: R_c, $\times$: R_s. Vertical lines are error bars.

§4. Case of $J_0 J_1 \neq 0$

The optimal conditions of f with respect to $\tilde{\Theta}_c$ and $\tilde{\Theta}_s$ are given by

$$\frac{\partial f}{\partial \tilde{\Theta}_c} = 0 : \tag{37}$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{I_1(\beta\Xi)}{I_0(\beta\Xi)}\frac{1}{\Xi}[-J_1^2 R_c R_s \sin(\tilde{\Theta}_c - \tilde{\Theta}_s)\sin\theta\cos\theta - J_0 J_1 R R_c \sin\tilde{\Theta}_c \cos\theta] = 0 \tag{38}$$

$$\frac{\partial f}{\partial \tilde{\Theta}_s} = 0 : \tag{39}$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{I_1(\beta\Xi)}{I_0(\beta\Xi)}\frac{1}{\Xi}[J_1^2 R_c R_s \sin(\tilde{\Theta}_c - \tilde{\Theta}_s)\sin\theta\cos\theta - J_0 J_1 R R_s \sin\tilde{\Theta}_s \sin\theta] = 0. \tag{40}$$

By adding eqs. (38) and (40), we obtain

$$R\int_0^{2\pi} d\theta \frac{I_1(\beta\Xi)}{I_0(\beta\Xi)}\frac{1}{\Xi}(R_c \sin\tilde{\Theta}_c \cos\theta + R_s \sin\tilde{\Theta}_s \sin\theta) = 0. \tag{41}$$

Defining $\tilde{R}$ and $\overline{\theta}$ as

$$R_c \sin\tilde{\Theta}_c \cos\theta + R_s \sin\tilde{\Theta}_s \sin\theta = \tilde{R}\cos(\theta - \overline{\theta}), \tag{42}$$

$$\tilde{R}\cos\overline{\theta} = R_c \sin\tilde{\Theta}_c,\ \tilde{R}\sin\overline{\theta} = R_s \sin\tilde{\Theta}_s, \tag{43}$$

$$\tilde{R} = \sqrt{(R_c \sin\tilde{\Theta}_c)^2 + (R_s \sin\tilde{\Theta}_s)^2}, \tag{44}$$

we obtain

$$\tilde{R}\int_0^{2\pi} d\theta \frac{I_1(\beta\Xi)}{I_0(\beta\Xi)}\frac{1}{\Xi}\cos(\theta - \overline{\theta}) = 0. \tag{45}$$

We define $\theta' = \frac{\pi}{2} - (\theta - \overline{\theta})$ and $\tilde{\Xi}(\theta') = \Xi(\overline{\theta} + \frac{\pi}{2} - \theta')$. Thus, by changing the integral range from $[0, 2\pi]$ to $[-\pi, \pi]$, eq. (45) reduces to

$$\tilde{R}\int_{-\pi}^{\pi} d\theta \frac{I_1(\beta\tilde{\Xi}(\theta))}{I_0(\beta\tilde{\Xi}(\theta))}\frac{1}{\tilde{\Xi}(\theta)}\sin\theta = 0. \tag{46}$$

Below, firstly, we consider the case of $\tilde{R} \neq 0$ and then the case of $\tilde{R} = 0$.

4.1 Solutions for $\tilde{R} \neq 0$

The necessary and sufficient condition for eq. (46) is that $\tilde{\Xi}(\theta)$ does not have the term $\sin\theta$. Since $\tilde{\Xi}(\theta)^2$ is rewritten as

$$\begin{aligned} \tilde{\Xi}(\theta)^2 \quad &= \\ [J_0 R + \frac{J_1}{\tilde{R}}\{(R_c\cos\tilde{\Theta}_c\sin\tilde{\Theta}_c + R_s\cos\tilde{\Theta}_s\sin\tilde{\Theta}_s)\sin\theta - R_c R_s\sin(\tilde{\Theta}_c - \tilde{\Theta}_s)\cos\theta\}]^2 & \\ + J_1^2\tilde{R}^2\sin^2\theta, & \end{aligned} \tag{47}$$

the condition is

$$R_c\sin 2\tilde{\Theta}_c + R_s\sin 2\tilde{\Theta}_s = 0. \tag{48}$$

Thus, the necessary and sufficient condition for eq. (48) is as follows:

(1) Case of $R_c R_s \neq 0$.

$$\sin 2\tilde{\Theta}_c = 0 \text{ and } \sin 2\tilde{\Theta}_s = 0.$$

That is,

$$\{\tilde{\Theta}_c = (0, \pm\frac{\pi}{2}, \pi) \ (\bmod 2\pi)\} \text{ and } \{\tilde{\Theta}_s = (0, \pm\frac{\pi}{2}, \pi) \ (\bmod 2\pi)\}.$$

Hereafter, we omit 'mod 2π' for simplicity.

(2) Case of $R_c = 0$.

$$R_s \neq 0 \text{ and } \{\tilde{\Theta}_s = (0, \pm\frac{\pi}{2}, \pi)\}.$$

(3) Case of $R_s = 0$.

$$R_c \neq 0 \text{ and } \{\tilde{\Theta}_c = (0, \pm\frac{\pi}{2}, \pi)\}.$$

Now, let us find the solution in each case. To simplify the descriptions, we introduce the following notations:

$$\langle g(\theta)\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{I_1(\beta\tilde{\Xi}(\theta))}{I_0(\beta\tilde{\Xi}(\theta))}\frac{1}{\tilde{\Xi}(\theta)}g(\theta) \tag{49}$$

$$\begin{aligned} \tilde{\Xi}(\theta) &= \Xi(\overline{\theta} + \frac{\pi}{2} - \theta) \\ &= \sqrt{\left(J_0 R - \frac{J_1}{\tilde{R}}R_c R_s\sin(\tilde{\Theta}_c - \tilde{\Theta}_s)\cos\theta\right)^2 + J_1^2\tilde{R}^2\sin^2\theta}. \end{aligned} \tag{50}$$

Thus, eqs. (38) and (40) become

$$\langle -J_1^2 R_c R_s \sin(\tilde{\Theta}_c - \tilde{\Theta}_s) \sin 2(\theta - \overline{\theta}) - 2J_0 J_1 R R_c \sin\tilde{\Theta}_c \sin(\theta - \overline{\theta})\rangle = 0, \tag{51}$$

$$\langle J_1^2 R_c R_s \sin(\tilde{\Theta}_c - \tilde{\Theta}_s) \sin(2\theta - \overline{\theta}) - 2J_0 J_1 R R_s \sin\tilde{\Theta}_s \cos(\theta - \overline{\theta})\rangle = 0. \tag{52}$$

These equations (51) and (52) reduce to the same equation as

$$\begin{aligned} &J_1^2 R_c R_s \sin(\tilde{\Theta}_c - \tilde{\Theta}_s) \sin(2\overline{\theta}) \langle \cos(2\theta)\rangle \\ &\qquad + \frac{2}{\tilde{R}} J_0 J_1 R R_c \sin\tilde{\Theta}_c R_s \sin\tilde{\Theta}_s \langle \cos\theta\rangle = 0. \end{aligned} \tag{53}$$

Here, we summarize the results of analysis of eq. (53). See Appendix A.1 for the derivation.
Solution 1. Case of $(\tilde{\Theta}_c, \tilde{\Theta}_s) = (\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$

$$\Xi(\theta) = \sqrt{(J_0 R)^2 + (J_1 R_1 \sin\theta)^2}, \tag{54}$$

$$f = \frac{1}{2}(J_0 R^2 + J_1 R_1^2) - \frac{1}{\beta}\frac{2}{\pi}\int_0^{\pi/2} d\theta \ln\{2\pi I_0(\beta\tilde{\Xi}(\theta))\}, \tag{55}$$

$$R = R J_0 \frac{2}{\pi}\int_0^{\pi/2} d\theta \frac{I_1(\beta\tilde{\Xi})}{I_0(\beta\tilde{\Xi})}\frac{1}{\tilde{\Xi}}, \tag{56}$$

$$R_1 = R_1 J_1 \frac{2}{\pi}\int_0^{\pi/2} d\theta \frac{I_1(\beta\tilde{\Xi})}{I_0(\beta\tilde{\Xi})}\frac{1}{\tilde{\Xi}}\sin^2(\theta). \tag{57}$$

From numerical results, this solution turns out to be the stable Pn solution.
Solution 2. Case of $(\tilde{\Theta}_c, \tilde{\Theta}_s) = (0, \frac{\pi}{2})$ and solution 3. Case of $(\tilde{\Theta}_c, \tilde{\Theta}_s) = (0, -\frac{\pi}{2})$

$$\tilde{\Xi}(\theta) = \sqrt{(J_0 R + J_1 R_c \cos\theta)^2 + (J_1 R_s \sin\theta)^2}. \tag{58}$$

When $R = 0$, this solution gives the spinning solution of $J_0 = 0$.
Solution 4. Case of $(\tilde{\Theta}_c, \tilde{\Theta}_s) = (\frac{\pi}{2}, 0)$ and solution 5. Case of $(\tilde{\Theta}_c, \tilde{\Theta}_s) = (-\frac{\pi}{2}, 0)$

$$\tilde{R} = R_c, \overline{\theta} = 0, \tag{59}$$

$$\tilde{\Xi}(\theta) = \sqrt{(J_0 R - J_1 R_s \cos\theta)^2 + (J_1 R_c \sin\theta)^2}. \tag{60}$$

If we put $\theta = \pi/2 - \theta'$, we have $\hat{\Xi}(\theta') \equiv \tilde{\Xi}(\pi/2 - \theta') = \sqrt{(J_0 R - J_1 R_s \sin\theta')^2 + (J_1 R_c \cos\theta')^2}$. Then, when $R = 0$, this gives the spinning solution of $J_0 = 0$.

These solutions except for solution 1 are unstable, which we will investigate later.

The solutions for the case of $\tilde{R} = 0$ are derived from the solutions for the case of $\tilde{R} \neq 0$. See Appendix A.2.

4.2 Analysis of SPEs (56) and (57), stable Pn solution

In this section, we analyze solution 1, which is the stable Pn solution.

4.2.1 Phase transition points

SPEs are

$$R = RJ_0 \frac{2}{\pi} \int_0^{\pi/2} d\theta \frac{I_1(\beta\tilde{\Xi})}{I_0(\beta\tilde{\Xi})} \frac{1}{\tilde{\Xi}}, \tag{61}$$

$$R_1 = R_1 J_1 \frac{2}{\pi} \int_0^{\pi/2} d\theta \frac{I_1(\beta\tilde{\Xi})}{I_0(\beta\tilde{\Xi})} \frac{1}{\tilde{\Xi}} \sin^2\theta, \tag{62}$$

$$\tilde{\Xi} = \sqrt{(J_0 R)^2 + (J_1 R_1 \sin\theta)^2}. \tag{63}$$

When $R \ll 1$ and $R_1 \ll 1$, $\tilde{\Xi} \ll 1$ follows, and then we have

$$I_0(\beta\tilde{\Xi}) \simeq \frac{1}{2\pi} \int_0^{2\pi} d\phi \left(1 + \beta\tilde{\Xi}\cos\phi + \frac{1}{2}(\beta\tilde{\Xi})^2(\cos\phi)^2\right) = 1 + \frac{1}{4}(\beta\tilde{\Xi})^2, \tag{64}$$

$$I_1(\beta\tilde{\Xi}) \simeq \frac{1}{2\pi} \int_0^{2\pi} d\phi \cos\phi \left(1 + \beta\tilde{\Xi}\cos\phi + \frac{1}{2}(\beta\tilde{\Xi})^2(\cos\phi)^2\right) = \frac{1}{2}\beta\tilde{\Xi}, \tag{65}$$

$$\frac{I_1}{I_0} \simeq \frac{1}{2}\beta\tilde{\Xi}. \tag{66}$$

Thus, SPEs become

$$R \simeq RJ_0 \frac{2}{\pi} \int_0^{\pi/2} d\theta \frac{1}{2}\beta = RJ_0 \frac{1}{2}\beta. \tag{67}$$

$$R_1 \simeq R_1 J_1 \frac{2}{\pi} \int_0^{\pi/2} d\theta \frac{1}{2}\beta \sin^2(\theta) = R_1 J_1 \frac{1}{4}\beta. \tag{68}$$

Therefore, we derive

$$T_{0,c} = \frac{J_0}{2}, \tag{69}$$

$$T_{1,c} = \frac{J_1}{4}. \tag{70}$$

The former is the critical temperature for the U solution and the latter is that for the S solution.

4.2.2 SPEs for $T \to 0$

When $T \ll 1$ ($\beta \gg 1$), we have

$$I_0(\beta\tilde{\Xi}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi e^{\beta\tilde{\Xi}\cos\phi} \simeq \frac{1}{2\pi} \int_{-\infty}^{\infty} d\phi e^{\beta\tilde{\Xi}\left(1-\frac{1}{2}\phi^2\right)} = \frac{1}{2\pi}\sqrt{\frac{2\pi}{\beta\tilde{\Xi}}} e^{\beta\tilde{\Xi}}, \tag{71}$$

$$I_1(\beta\tilde{\Xi}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \cos\phi e^{\beta\tilde{\Xi}\cos\phi} \simeq \frac{1}{2\pi} \int_{-\infty}^{\infty} d\phi e^{\beta\tilde{\Xi}\left(1-\frac{1}{2}\phi^2\right)} \simeq I_0(\beta\tilde{\Xi}). \tag{72}$$

Therefore, $\frac{I_1(\beta\tilde{\Xi})}{I_0(\beta\tilde{\Xi})} \simeq 1$ follows. Thus, the SPEs are

$$R \simeq RJ_0 \frac{2}{\pi} \int_0^{\pi/2} d\theta \frac{1}{\tilde{\Xi}}, \tag{73}$$

$$R_1 \simeq R_1 J_1 \frac{2}{\pi} \int_0^{\pi/2} d\theta \frac{1}{\tilde{\Xi}} \sin^2\theta. \tag{74}$$

Let us define

$$
\begin{aligned}
k &\equiv \frac{J_1 R_1}{J_0 R} = J\frac{R_1}{R}, &(75)\\
J &\equiv \frac{J_1}{J_0}. &(76)
\end{aligned}
$$

Then, $\tilde{\Xi} = J_0 R\sqrt{1+k^2\sin^2\theta}$ follows. Thus, we have

$$
\begin{aligned}
R &\simeq \frac{2}{\pi}\int_0^{\pi/2} d\theta \frac{1}{\sqrt{1+k^2\sin^2\theta}}, &(77)\\
R_1 &\simeq k\frac{2}{\pi}\int_0^{\pi/2} d\theta \frac{1}{\sqrt{1+k^2\sin^2\theta}}\sin^2\theta. &(78)
\end{aligned}
$$

4.2.3 Appearance of Pn solution when $T \to 0$

Let us assume $0 < k \ll 1$. Then, we have

$$
\begin{aligned}
R &\simeq \frac{2}{\pi}\int_0^{\pi/2} d\theta \frac{1}{\sqrt{1+k^2\sin^2\theta}} \simeq \frac{2}{\pi}\int_0^{\pi/2} d\theta \left(1-\frac{k^2}{2}\sin^2\theta\right) = 1-\frac{k^2}{4}, &(79)\\
R_1 &\simeq k\frac{2}{\pi}\int_0^{\pi/2} d\theta \left(1-\frac{k^2}{2}\sin^2\theta\right)\sin^2\theta = k(\frac{1}{2}-\frac{3k^2}{16}). &(80)
\end{aligned}
$$

By using $R_1 = kR/J$, we obtain

$$
\begin{aligned}
R &= \frac{J}{3J-4}, &(81)\\
R_1 &= \frac{kR}{J} = \frac{2\sqrt{2}\sqrt{J-2}}{(3J-4)^{3/2}}, &(82)\\
k^2 &= \frac{8(J-2)}{3J-4}. &(83)
\end{aligned}
$$

Therefore, the Pn solution emerges for $J > 2$, that is, for $J_1 > 2J_0$. From this analysis, it turns out that the Pn solution bifurcates from the U solution.

4.2.4 Appearance of Pn solution at finite temperatures

Let us assume $R > 0$ and $R_1 \ll 1$. Then, $k \ll 1$ follows. Since $\tilde{\Xi}$ is expressed as

$$
\tilde{\Xi} = J_0 R\sqrt{1+k^2\sin^2\theta} \simeq J_0 R(1+\frac{1}{2}k^2\sin^2\theta), \qquad (84)
$$

we have

$$
\begin{aligned}
I_0(\beta\tilde{\Xi}) &= \frac{1}{2\pi}\int_0^{2\pi} d\phi e^{\beta\tilde{\Xi}\cos\phi} \simeq \frac{1}{2\pi}\int_0^{2\pi} d\phi e^{\beta J_0 R\cos\phi} e^{\beta J_0 R\frac{1}{2}k^2\sin^2\theta\cos\phi} \\
&\simeq \frac{1}{2\pi}\int_0^{2\pi} d\phi e^{\beta J_0 R\cos\phi}(1+\beta J_0 R\frac{1}{2}k^2\sin^2\theta\cos\phi) \\
&= I_0(\beta J_0 R) + \beta J_0 R\frac{1}{2}k^2\sin^2\theta I_1(\beta J_0 R) \qquad (85) \\
I_1(\beta\tilde{\Xi}) &\simeq I_1(\beta J_0 R) + \beta J_0 R\frac{1}{2}k^2\sin^2\theta I_2(\beta J_0 R), \qquad (86) \\
\frac{I_1(\beta\tilde{\Xi})}{I_0(\beta\tilde{\Xi})} &= \frac{I_1(\beta J_0 R)+\mathcal{O}(k^2)}{I_0(\beta J_0 R)+\mathcal{O}(k^2)} \simeq \frac{I_1(\beta J_0 R)}{I_0(\beta J_0 R)}. \qquad (87)
\end{aligned}
$$

Therefore, the SPE for R becomes

$$
\begin{aligned}
R &\simeq RJ_0\frac{2}{\pi}\int_0^{\pi/2} d\theta \frac{I_1(\beta\tilde{\Xi})}{I_0(\beta\tilde{\Xi})}\frac{1}{\tilde{\Xi}}\frac{1}{J_0 R}(1-\frac{k^2}{2}\sin^2\theta) \\
&\simeq \frac{2}{\pi}\int_0^{\pi/2} d\theta\frac{I_1(\beta J_0 R)}{I_0(\beta J_0 R)} = \frac{I_1(\beta J_0 R)}{I_0(\beta J_0 R)}. \qquad (88)
\end{aligned}
$$

This is the equation for R when $J_1 = 0$, that is, this is the equation for the U solution. The equation for R_1 is

$$
\begin{aligned}
R_1 &= R_1 J_1\frac{2}{\pi}\int_0^{\pi/2} d\theta\frac{I_1(\beta\tilde{\Xi})}{I_0(\beta\tilde{\Xi})}\frac{1}{\tilde{\Xi}}\sin^2(\theta) \simeq k\frac{2}{\pi}\int_0^{\pi/2} d\theta\frac{I_1(\beta\tilde{\Xi})}{I_0(\beta\tilde{\Xi})}(1-\frac{k^2}{2}\sin^2\theta) \\
&\simeq k\frac{2}{\pi}\int_0^{\pi/2} d\theta\frac{I_1(\beta J_0 R)}{I_0(\beta J_0 R)}\sin^2\theta = \frac{kR}{2}. \qquad (89)
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
R_1 &= \frac{kR}{2} = \frac{J_1 R_1}{J_0 R}\frac{R}{2} = \frac{J_1 R_1}{2J_0}, \\
1 &= \frac{J_1}{2J_0}, \\
J_c &= 2 = \left(\frac{J_1}{J_0}\right)_c. \qquad (90)
\end{aligned}
$$

Thus, it turns out that the Pn solution bifurcates from the U solution at $J_c = 2$ in the finite temperature as well.

4.2.5 Numerical results

In Fig. 2, we display the temperature dependences of order parameters for $J_0 = 1$ and $J_1 = 2.1$. Theoretical results are obtained by numerically solving the SPEs (56) and (57) for the Pn solution, and the SPE (35) for the S solution. The theoretical and numerical results agree well.

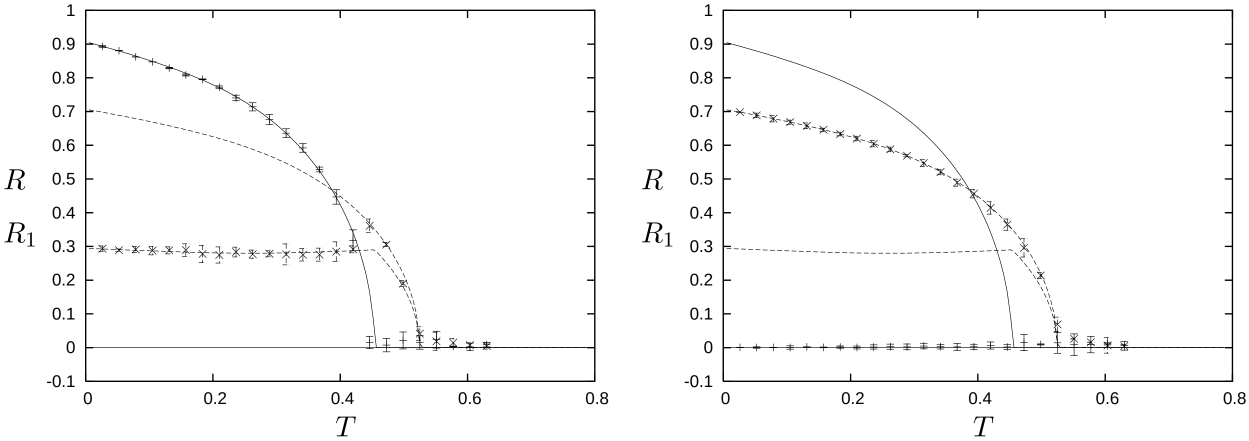

Fig. 2. Temperature dependences of order parameters. Curves: theory; eq. (35) for the U solution and eqs. (56) and (57) for the Pn solution. Symbols: Monte Carlo simulation ($N = 10000$). Solid curve and $+$: R. Dashed curve and $\times$: R_1. In (a) and (b), theoretical results for the S and Pn solutions are depicted. Symbols: (a) Pn solution ($R > 0, R_1 > 0$), (b) S solution ($R = 0, R_1 > 0$). In (a), since the Pn solution disappears at higher temperatures, the S solution is numerically obtained at those temperatures.

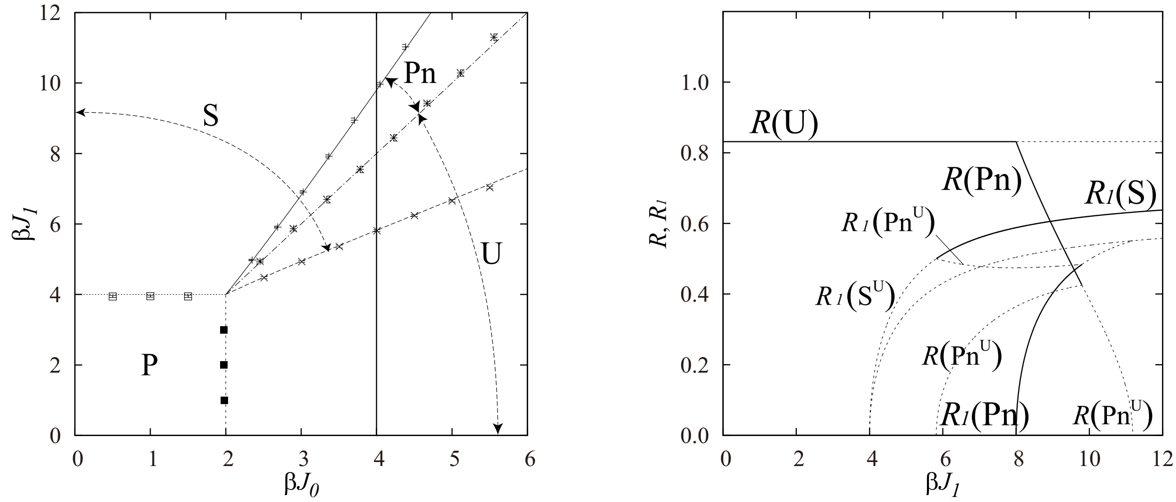

Fig. 3. (a) phase diagram of the scaled parameter space. Curves: theory. Symbols: Monte Carlo simulation. The vertical line is the parameter shown in (b). (b) βJ_1 dependences of order parameters in the XY model. $\beta J_0 = 4$. Solid curves: stable solutions; dashed curves: unstable solutions with superscript U, e.g., S^U.

§5. Determination of Phase Boundaries of Bistable Regions

In this section, we study the boundaries of several phases in (J_0, J_1) space. We showed that the boundary between the U and Pn phases is given by $J_1 = 2J_0$ in the previous section. In this section, we study the boundary between the S and U phases and that between the Pn and S phases.

5.1 Boundary between the S and U phases

As noted in Ref. 7, when βJ_1 is reduced by fixing βJ_0 to 4, an unstable Pn solution and a stable S solution merge, and an unstable S solution appears. See Fig. 3(b). At the parameter where the stable U solution disappears, the R of the Pn solution is 0, and the R_1 values of the Pn and S solutions are the same. Before and after the disappearance of the stable S solution, there exists a stable U solution. Thus, the boundary between the S and U solutions is where the stable S solution disappears. Solutions 2 and 4 for the Pn solution coincide when $R_c = R_s$, and these solutions give the spinning solutions when $R = 0$. Therefore, it is considered that solution 2 is the unstable Pn solution. We do not assume $R_c = R_s$, but it is proved that this relation holds at the boundary. The quantities we treat are

$$f = \frac{1}{2}(J_0 R^2 + J_1 R_1^2) - \frac{1}{\beta}\frac{1}{\pi}\int_0^{\pi} d\theta \ln\{2\pi I_0(\beta\tilde{\Xi}(\theta))\}, \tag{91}$$

$$\tilde{\Xi}(\theta) = \sqrt{(J_0 R + J_1 R_c \cos\theta)^2 + (J_1 R_s \sin\theta)^2}. \tag{92}$$

The SPEs are

$$R = \langle J_0 R + J_1 R_c \cos\theta \rangle, \tag{93}$$

$$R_c = \langle (J_0 R + J_1 R_c \cos\theta)\cos\theta \rangle, \tag{94}$$

$$R_s = J_1 R_s \langle \sin^2\theta \rangle, \tag{95}$$

$$\langle A \rangle = \frac{1}{\pi}\int_0^{\pi} d\theta \frac{I_1(\beta\tilde{\Xi}(\theta))}{I_0(\beta\tilde{\Xi}(\theta))}\frac{1}{\tilde{\Xi}(\theta)} A, \tag{96}$$

$$I_n(z) = \frac{1}{2\pi}\int_0^{2\pi} d\phi e^{z\cos\phi}\cos(n\phi). \tag{97}$$

Assuming $R \ll 1$, the Taylor expansion of $\tilde{\Xi}(\theta)$ up to $\mathcal{O}(R)$ becomes

$$\tilde{\Xi}(\theta) \simeq J_1 \tilde{\Xi}_0(\theta)\Big(1 + \frac{J_0 R_c \cos\theta}{J_1 \tilde{\Xi}_0(\theta)^2} R\Big), \tag{98}$$

$$\tilde{\Xi}_0(\theta) = \sqrt{(R_c\cos\theta)^2 + (R_s \sin\theta)^2}. \tag{99}$$

Therefore, I_0 and I_1 are expressed as

$$\begin{aligned}
I_0(\beta\tilde{\Xi}(\theta)) &\simeq \frac{1}{2\pi}\int_0^{2\pi} d\phi e^{\beta J_1\tilde{\Xi}_0(\theta)\cos\phi}\Big(1+\frac{\beta J_0 R_c\cos\theta}{\tilde{\Xi}_0(\theta)}R\cos\phi\Big)\\
&= I_0(\beta J_1\tilde{\Xi}_0(\theta)) + I_1(\beta J_1\tilde{\Xi}_0(\theta))\frac{\beta J_0 R_c\cos\theta}{\tilde{\Xi}_0(\theta)}R,\\
I_1(\beta\tilde{\Xi}(\theta)) &\simeq \frac{1}{2\pi}\int_0^{2\pi} d\phi e^{\beta J_1\tilde{\Xi}_0(\theta)\cos\phi}\Big(1+\frac{\beta J_0 R_c\cos\theta}{\tilde{\Xi}_0(\theta)}R\cos\phi\Big)\cos\phi\\
&= I_1(\beta J_1\tilde{\Xi}_0(\theta)) + \frac{1}{2}\Big(I_0(\beta J_1\tilde{\Xi}_0(\theta)) + I_2(\beta J_1\tilde{\Xi}_0(\theta))\Big)\frac{\beta J_0 R_c\cos\theta}{\tilde{\Xi}_0(\theta)}R.
\end{aligned}$$

Thus, the SPEs are

$$R \simeq J_0 R\langle 1\rangle_0 + J_1 R_c\langle\cos\theta\rangle, \tag{100}$$

$$R_c \simeq J_0 R\langle\cos\theta\rangle_0 + J_1 R_c\langle\cos^2\theta\rangle, \tag{101}$$

$$R_s \simeq J_1 R_s\langle\sin^2\theta\rangle, \tag{102}$$

$$\langle A\rangle_0 = \frac{1}{\pi}\int_0^{\pi} d\theta\frac{I_1(\beta J_1\tilde{\Xi}_0(\theta))}{I_0(\beta J_1\tilde{\Xi}_0(\theta))}\frac{1}{J_1\tilde{\Xi}_0(\theta)}A. \tag{103}$$

Since $\langle\cos\theta\rangle_0 = 0$, taking the limit $R\to 0$ in eqs. (101) and (102), $\langle\cos^2\theta\rangle_0 = \langle\sin^2\theta\rangle_0$ follows exactly when $R_cR_s \neq 0$. This implies $R_c = R_s$ at the phase boundary. Thus, we have the following relations:

$$\begin{aligned}
\tilde{\Xi}_0(\theta) &= R_c = \frac{R_1}{\sqrt{2}},\\
\tilde{\Xi}(\theta) &\simeq J_1 R_c + J_0 R\cos\theta,\\
I_0(\beta\tilde{\Xi}(\theta)) &\simeq I_0(\bar{J}_1 R_c) + I_1(\bar{J}_1 R_c)\bar{J}_0 R\cos\theta,\\
I_1(\beta\tilde{\Xi}(\theta)) &\simeq I_1(\bar{J}_1 R_c) + \frac{1}{2}\Big(I_0(\bar{J}_1 R_c) + I_2(\bar{J}_1 R_c)\Big)\bar{J}_0 R\cos\theta,\\
\frac{I_1(\beta\tilde{\Xi}(\theta))}{I_0(\beta\tilde{\Xi}(\theta))} &\simeq \frac{I_1(\bar{J}_1 R_c)}{I_0(\bar{J}_1 R_c)}\frac{1+\frac{1}{2}\Big(\frac{I_0(\bar{J}_1 R_c)+I_2(\bar{J}_1 R_c)}{I_1(\bar{J}_1 R_c)}\Big)\bar{J}_0 R\cos\theta}{1+\bar{J}_0 R\cos\theta\frac{I_1(\bar{J}_1 R_c)}{I_0(\bar{J}_1 R_c)}}\\
&\simeq \frac{I_1(\bar{J}_1 R_c)}{I_0(\bar{J}_1 R_c)}[1+\Big(\frac{I_0(\bar{J}_1 R_c)+I_2(\bar{J}_1 R_c)}{2I_1(\bar{J}_1 R_c)} - \frac{I_1(\bar{J}_1 R_c)}{I_0(\bar{J}_1 R_c)}\Big)\bar{J}_0 R\cos\theta],\\
\frac{1}{\tilde{\Xi}(\theta)} &\simeq \frac{1}{J_1 R_c(1+\frac{J_0 R}{J_1 R_c}\cos\theta)} \simeq \frac{1}{J_1 R_c}(1-\frac{J_0 R}{J_1 R_c}\cos\theta),\\
\frac{I_1(\beta\tilde{\Xi}(\theta))}{I_0(\beta\tilde{\Xi}(\theta))}\frac{1}{\tilde{\Xi}(\theta)} &\simeq \frac{I_1(\bar{J}_1 R_c)}{I_0(\bar{J}_1 R_c)}\frac{1}{J_1 R_c}\\
&\times[1+\Big(\frac{I_0(\bar{J}_1 R_c)+I_2(\bar{J}_1 R_c)}{2I_1(\bar{J}_1 R_c)} - \frac{I_1(\bar{J}_1 R_c)}{I_0(\bar{J}_1 R_c)} - \frac{1}{\bar{J}_1 R_c}\Big)\bar{J}_0 R\cos\theta.]
\end{aligned}$$

Here, we put $\bar{J}_n = \beta J_n$. By using these relations, $\langle \cos\theta \rangle$ is expressed as

$$\begin{aligned}\langle \cos\theta \rangle &= \frac{1}{\pi}\int_0^\pi d\theta \frac{I_1(\beta\tilde{\Xi}(\theta))}{I_0(\beta\tilde{\Xi}(\theta))}\frac{1}{\tilde{\Xi}(\theta)}\cos\theta \\ &\simeq \frac{1}{\pi}\int_0^\pi d\theta \frac{I_1(\bar{J}_1 R_c)}{I_0(\bar{J}_1 R_c)}\frac{1}{J_1 R_c} \\ &\quad \times[\cos\theta + \Big(\frac{I_0(\bar{J}_1 R_c)+I_2(\bar{J}_1 R_c)}{2I_1(\bar{J}_1 R_c)} - \frac{I_1(\bar{J}_1 R_c)}{I_0(\bar{J}_1 R_c)} - \frac{1}{\bar{J}_1 R_c}\Big)\bar{J}_0 R\cos^2\theta] \\ &= \frac{I_1(\bar{J}_1 R_c)}{2I_0(\bar{J}_1 R_c)}\frac{1}{J_1 R_c}\Big(\frac{I_0(\bar{J}_1 R_c)+I_2(\bar{J}_1 R_c)}{2I_1(\bar{J}_1 R_c)} - \frac{I_1(\bar{J}_1 R_c)}{I_0(\bar{J}_1 R_c)} - \frac{1}{\bar{J}_1 R_c}\Big)\bar{J}_0 R.\end{aligned}$$

Below, we put $\bar{I}_n = I_n(\bar{J}_1 R_c)$. Equation (100) becomes

$$\begin{aligned}R &\simeq J_0 R\frac{\bar{I}_1}{J_1 R_c \bar{I}_0} + J_1 R_c \frac{\bar{I}_1}{2\bar{I}_0}\frac{1}{J_1 R_c}\Big(\frac{\bar{I}_0+\bar{I}_2}{2\bar{I}_1} - \frac{\bar{I}_1}{\bar{I}_0} - \frac{1}{\bar{J}_1 R_c}\Big)\bar{J}_0 R \\ &= \bar{J}_0 R\frac{\bar{I}_1}{\bar{J}_1 R_c \bar{I}_0} + \frac{\bar{I}_1}{2\bar{I}_0}\Big(\frac{\bar{I}_0+\bar{I}_2}{2\bar{I}_1} - \frac{\bar{I}_1}{\bar{I}_0} - \frac{1}{\bar{J}_1 R_c}\Big)\bar{J}_0 R. \end{aligned} \tag{104}$$

Therefore, the following relation holds at the boundary:

$$1 = \bar{J}_0\frac{\bar{I}_1}{\bar{J}_1 R_c \bar{I}_0} + \frac{\bar{I}_1}{2\bar{I}_0}\Big(\frac{\bar{I}_0+\bar{I}_2}{2\bar{I}_1} - \frac{\bar{I}_1}{\bar{I}_0} - \frac{1}{\bar{J}_1 R_c}\Big)\bar{J}_0. \tag{105}$$

On the other hand, at the boundary, eq. (101) becomes

$$\begin{aligned}1 &= J_1\langle\cos^2\theta\rangle_0 \\ &= J_1\frac{1}{\pi}\int_0^\pi d\theta\frac{\bar{I}_1}{\bar{I}_0}\frac{1}{J_1 R_c}\cos^2\theta = J_1\frac{\bar{I}_1}{\bar{I}_0}\frac{1}{2J_1 R_c} = \frac{\bar{I}_1}{\bar{I}_0}\frac{1}{2R_c}.\end{aligned}$$

Therefore, we obtain

$$R_c = \frac{\bar{I}_1}{2\bar{I}_0}. \tag{106}$$

This is nothing but the equation for the stable S solution. Thus, eq. (105) becomes

$$\begin{aligned}1 &= \bar{J}_0\frac{2}{\bar{J}_1} + R_c\Big(\frac{\bar{I}_0+\bar{I}_2}{2\bar{I}_1} - \frac{\bar{I}_1}{\bar{I}_0} - \frac{1}{\bar{J}_1 R_c}\Big)\bar{J}_0 \\ &= \frac{\bar{J}_0}{\bar{J}_1} + R_c\Big(\frac{\bar{I}_0+\bar{I}_2}{2\bar{I}_1} - \frac{\bar{I}_1}{\bar{I}_0}\Big)\bar{J}_0.\end{aligned}$$

Therefore, the equation which determines the boundary between the S and U phases is given by

$$\bar{J}_0 = [\frac{1}{\bar{J}_1} + R_c\Big(\frac{\bar{I}_0+\bar{I}_2}{2\bar{I}_1} - \frac{\bar{I}_1}{\bar{I}_0}\Big)]^{-1} = [\frac{1}{\bar{J}_1} + \Big(\frac{\bar{I}_0+\bar{I}_2}{2\bar{I}_1} - \frac{\bar{I}_1}{\bar{I}_0}\Big)\frac{\bar{I}_1}{2\bar{I}_0}]^{-1}. \tag{107}$$

5.2 Boundary between the S and Pn phases

As studied in Ref. 7, when βJ_1 is increased by fixing βJ_0 to 4, a stable Pn solution and an unstable Pn solution merge and only the unstable Pn solution remains. See Fig. 3(b). At the parameter where the stable Pn solution disappears, $R_c = 0$ holds. The unstable Pn solution is solution 2 and it coincides with solution 1 when $R_c = 0$. Therefore, the boundary between the S and Pn phases is where R_c becomes 0 for solution 2. Assuming $R_c \ll 1$ in solution 2, the Taylor expansion of $\hat{\Xi}(\theta) = \beta\tilde{\Xi}$ up to $\mathcal{O}(R_c)$ is

$$\begin{aligned}\hat{\Xi}(\theta) &= \sqrt{(\bar{J}_0 R + \bar{J}_1 R_c \cos\theta)^2 + (\bar{J}_1 R_s \sin\theta)^2} \\ &\simeq \sqrt{(\bar{J}_0 R)^2 + (\bar{J}_1 R_s \sin\theta)^2 + 2\bar{J}_0\bar{J}_1 R R_c \cos\theta} \\ &\simeq \hat{\Xi}_0\Big(1 + \frac{\bar{J}_0\bar{J}_1 R R_c \cos\theta}{\hat{\Xi}_0^2}\Big), \qquad (108)\\ \hat{\Xi}_0(\theta) &= \sqrt{(\bar{J}_0 R)^2 + (\bar{J}_1 R_s \sin\theta)^2}. \qquad (109)\end{aligned}$$

Therefore, I_0 and I_1 are expressed as

$$\begin{aligned}I_0(\hat{\Xi}(\theta)) &\simeq \frac{1}{2\pi}\int_0^{2\pi} d\phi e^{\hat{\Xi}_0\cos\phi}\Big(1 + \frac{\bar{J}_0\bar{J}_1 R\cos\theta}{\hat{\Xi}_0(\theta)} R_c\cos\phi\Big) \\ &= I_0^* + \frac{\bar{J}_0\bar{J}_1 R\cos\theta}{\hat{\Xi}_0(\theta)} R_c I_1^*, \qquad (110)\\ I_1(\hat{\Xi}(\theta)) &\simeq I_1^* + \frac{\bar{J}_0\bar{J}_1 R\cos\theta}{\hat{\Xi}_0(\theta)} R_c \frac{I_0^* + I_2^*}{2}, \qquad (111)\\ \frac{I_1(\hat{\Xi}(\theta))}{I_0(\hat{\Xi}(\theta))}\frac{1}{\hat{\Xi}(\theta)} &\simeq \frac{I_1^*}{I_0^*}\frac{1}{\hat{\Xi}_0} \\ &\times[1 + \Big(\frac{I_0^* + I_2^*}{2I_1^*} - \frac{I_1^*}{I_0^*} - \frac{1}{\hat{\Xi}_0}\Big)\frac{1}{\hat{\Xi}_0}\bar{J}_0\bar{J}_1 R R_c\cos\theta]. \qquad (112)\end{aligned}$$

Here, we put $I_n^* = I_n(\hat{\Xi}_0)$. Therefore, the SPE for R_c, eq. (94), is

$$\begin{aligned}R_c &= \frac{1}{\pi}\int_0^{\pi} d\theta \frac{I_1(\hat{\Xi}(\theta))}{I_0(\hat{\Xi}(\theta))}\frac{1}{\hat{\Xi}(\theta)}(\bar{J}_0 R + \bar{J}_1 R_c\cos\theta)\cos\theta \\ &\simeq \frac{1}{\pi}\int_0^{\pi} d\theta \frac{I_1^*}{I_0^*}\frac{1}{\hat{\Xi}_0} \\ &\times[1 + \Big(\frac{I_0^* + I_2^*}{2I_1^*} - \frac{I_1^*}{I_0^*} - \frac{1}{\hat{\Xi}_0}\Big)\frac{1}{\hat{\Xi}_0}\bar{J}_0\bar{J}_1 R R_c\cos\theta](\bar{J}_0 R + \bar{J}_1 R_c\cos\theta)\cos\theta.\end{aligned}$$

Since the integration of odd power of $\cos\theta$ is 0, we obtain

$$\begin{aligned}R_c &\simeq \frac{1}{\pi}\int_0^{\pi} d\theta \frac{I_1^*}{I_0^*}\frac{1}{\hat{\Xi}_0} \\ &\times\bar{J}_1 R_c[1 + \Big(\frac{I_0^* + I_2^*}{2I_1^*} - \frac{I_1^*}{I_0^*} - \frac{1}{\hat{\Xi}_0}\Big)\frac{1}{\hat{\Xi}_0}(\bar{J}_0 R)^2]\cos^2\theta. \qquad (113)\end{aligned}$$

Therefore, the boundary between the S and Pn phases is determined by

$$1 = \frac{1}{\pi}\int_0^{\pi} d\theta \frac{I_1^*}{I_0^*}\frac{1}{\hat{\Xi}_0}\bar{J}_1[1 + \Big(\frac{I_0^* + I_2^*}{2I_1^*} - \frac{I_1^*}{I_0^*} - \frac{1}{\hat{\Xi}_0}\Big)\frac{1}{\hat{\Xi}_0}(\bar{J}_0 R)^2]\cos^2\theta. \qquad (114)$$

On the other hand, the SPEs for R, eq. (93), and R_s, eq. (95), at the boundary are

$$1 = \bar{J}_0\frac{1}{\pi}\int_0^{\pi} d\theta \frac{I_1^*}{I_0^*}\frac{1}{\hat{\Xi}_0}, \qquad (115)$$

$$1 = \bar{J}_1\frac{1}{\pi}\int_0^{\pi} d\theta \frac{I_1^*}{I_0^*}\frac{1}{\hat{\Xi}_0}\sin^2\theta. \qquad (116)$$

These are the SPEs for solution 1 of the Pn phase.

5.3 Numerical results

We display the phase diagram in the scaled parameter space in Fig. 3(a). There are five curves in the figure, and these curves represent theoretical results. Those are $\beta J_0 = 2$, eq. (18) for the boundary between the P and U phases, $\beta J_1 = 4$, eq. (36) for that between the P and S phases, $J_1 = 2J_0$, eq. (90) for that between the U and Pn phases, eq. (107) for that between the U and S phases, and eq. (114) for that between the S and Pn phases. The theoretical and numerical results agree well.

§6. Summary and Discussion

In this paper, we studied the XY model on a circle with the Mexican-hat type interaction. The interaction is composed of two terms, one of which is the uniform interaction with the strength J_0, and the other is the sinusoidal interaction with respect to the location θ of oscillators with the strength J_1. If $J_1 = 0$, it is the ferromagnetic XY model. The order parameters that characterize solutions of SPEs are R and R_1. The SPEs for the XY model were obtained analytically. There are four phases. The paramagnetic phase with $R = 0$ and $R_1 = 0$ is the disordered phase. In the uniform phase with $R > 0$ and $R_1 = 0$, the phases of the XY spins do not depend on the location of spins. In the spinning phase with $R = 0$ and $R_1 > 0$, the phases of the XY spins change by 2π when the coordinate of spin θ changes by 2π. In the pendulum phase with $R > 0$ and $R_1 > 0$, the phases do not change by 2π but fluctuate when the coordinate of spins θ changes by 2π.

The SPEs and the differences between the phases of complex order parameters were derived analytically. We proved that the Pn solution bifurcates from the U solution at $J_1 = 2J_0$ and found that the coexisting stable nontrivial solutions are the U and S solutions, and the S and Pn solutions. Finally, we derived the boundary between the S and U phases, and that between the S and Pn phases.

In the present study, we considered the first two Fourier components as the interaction components. How the existing phases and phase transitions depend on the types of the interaction is an interesting question. The XY model in which the interaction is composed of the first and second Fourier components is now under investigation, and different types of phases and phase transitions are found. These results will be reported elsewhere.

In the context of the synchronization - desynchronization phase transition, the phase oscillator network has been investigated extensively since Kuramoto introduced the globally coupled model.[2–4,8–12] In general, the phase oscillator model with uniform natural frequency coincides with the classical XY model at zero temperature, if the interactions in both models are the same. We are now studying the phase oscillator network model with the Mexican-hat type interaction and clarifying the resemblance of both models. These results will be reported in the future.

Appendix: Derivation of SPEs

A.1 Case of $\tilde{R} \neq 0$

(1) Case of $R_c R_s \neq 0$

Solution 1. Case of $\tilde{\Theta}_c = -\frac{\pi}{2}$ and $\tilde{\Theta}_s = \frac{\pi}{2}$.

From eqs. (42) – (44), we obtain

$$\tilde{R}\cos\overline{\theta} = -R_c, \quad \tilde{R}\sin\overline{\theta} = R_s, \tag{A·1}$$

$$\tilde{R} = R_1 = \sqrt{R_c^2 + R_s^2}, \tag{A·2}$$

$$\tilde{\Xi}(\theta) = \sqrt{(J_0 R)^2 + (J_1 R_1 \sin\theta)^2}. \tag{A·3}$$

Therefore, we obtain

$$f = \frac{1}{2}(J_0 R^2 + J_1 R_1^2) - \frac{1}{\beta}\frac{2}{\pi}\int_0^{\pi/2} d\theta \ln\{2\pi I_0(\beta\tilde{\Xi}(\theta))\}. \tag{A·4}$$

Since $\langle\cos\theta\rangle = 0$, eq. (53) is automatically satisfied. The SPEs are

$$R = RJ_0\frac{2}{\pi}\int_0^{\pi/2} d\theta \frac{I_1(\beta\tilde{\Xi})}{I_0(\beta\tilde{\Xi})}\frac{1}{\tilde{\Xi}}, \tag{A·5}$$

$$R_1 = R_1 J_1 \frac{2}{\pi}\int_0^{\pi/2} d\theta \frac{I_1(\beta\tilde{\Xi})}{I_0(\beta\tilde{\Xi})}\frac{1}{\tilde{\Xi}}\sin^2(\theta). \tag{A·6}$$

The solution for these equations agrees with numerical results. For the following cases of $(\tilde{\Theta}_c, \tilde{\Theta}_s)$,

$$(\tilde{\Theta}_c, \tilde{\Theta}_s) = (\frac{\pi}{2}, \frac{\pi}{2}), (\frac{\pi}{2}, -\frac{\pi}{2}), (-\frac{\pi}{2}, -\frac{\pi}{2}), \tag{A·7}$$

$\tilde{\Xi}$ is the same as eq. (A·3). Therefore, the SPEs for these cases are the same as (A·5) and (A·6).

For the following cases of $(\tilde{\Theta}_c, \tilde{\Theta}_s)$,

$$(\tilde{\Theta}_c, \tilde{\Theta}_s) = (0,0), (\pi,\pi), (0,\pi), (\pi,0), \tag{A·8}$$

from eq. (44), $\tilde{R} = 0$ follows, and these cases are excluded.

Solution 2. Case of $(\tilde{\Theta}_c, \tilde{\Theta}_s) = (0, \frac{\pi}{2})$

From eq. (44), $\tilde{R} = R_s$ follows. From eq. (43), we obtain $R_s\cos\overline{\theta} = 0, R_s\sin\overline{\theta} = R_s$. That is, $\overline{\theta} = \frac{\pi}{2}$ follows. Therefore, we obtain

$$\tilde{R} = R_s, \overline{\theta} = \frac{\pi}{2}, \tag{A·9}$$

$$\tilde{\Xi}(\theta) = \sqrt{(J_0 R + J_1 R_c \cos\theta)^2 + (J_1 R_s \sin\theta)^2}. \tag{A·10}$$

Thus, eq. (53) is automatically satisfied.

When $R=0$, this solution gives the spinning solution of $J_0=0$.

Solution 3 (= solution2). Case of $(\tilde{\Theta}_c,\tilde{\Theta}_s)=(0,-\frac{\pi}{2})$

From eq. (44), $\tilde{R}=R_s$ follows. From eq. (43), we obtain $R_s\cos\overline{\theta}=0, R_s\sin\overline{\theta}=-R_s$. That is, $\overline{\theta}=-\frac{\pi}{2}$ follows. Therefore, we obtain

$$\tilde{R}=R_s,\overline{\theta}=-\frac{\pi}{2}, \quad (A\cdot 11)$$

$$\tilde{\Xi}(\theta) = \sqrt{(J_0R-J_1R_c\cos\theta)^2+(J_1R_s\sin\theta)^2}. \quad (A\cdot 12)$$

Thus, eq. (53) is automatically satisfied. If we put $\theta'=\pi-\theta$, this case coincides with the case of $(\tilde{\Theta}_c,\tilde{\Theta}_s)=(0,\frac{\pi}{2})$.

Solution 4. Case of $(\tilde{\Theta}_c,\tilde{\Theta}_s)=(\frac{\pi}{2},0)$

From eq. (44), $\tilde{R}=R_c$ follows. From eq. (43), we obtain $R_c\cos\overline{\theta}=R_c, R_c\sin\overline{\theta}=0$. That is, $\overline{\theta}=0$ follows. Therefore, we obtain

$$\tilde{R}=R_c,\overline{\theta}=0, \quad (A\cdot 13)$$

$$\tilde{\Xi}(\theta) = \sqrt{(J_0R-J_1R_s\cos\theta)^2+(J_1R_c\sin\theta)^2}. \quad (A\cdot 14)$$

Thus, eq. (53) is automatically satisfied. If we put $\theta=\pi/2-\theta'$, we have $\hat{\Xi}(\theta')\equiv\tilde{\Xi}(\pi/2-\theta')=\sqrt{(J_0R-J_1R_s\sin\theta')^2+(J_1R_c\cos\theta')^2}$. Then, when $R=0$, this gives the spinning solution of $J_0=0$.

Solution 5 (=solution 4). Case of $(\tilde{\Theta}_c,\tilde{\Theta}_s)=(-\frac{\pi}{2},0)$

From eqs. (43) and (44), $\tilde{R}=-R_c$ and $\overline{\theta}=\pi$ follow. Therefore, we have

$$\tilde{R}=-R_c,\overline{\theta}=\pi, \quad (A\cdot 15)$$

$$\tilde{\Xi}(\theta) = \sqrt{(J_0R+J_1R_s\cos\theta)^2+(J_1R_c\sin\theta)^2}. \quad (A\cdot 16)$$

Thus, eq. (53) is automatically satisfied. Putting $\theta'=\pi-\theta$, we note that this case coincides with the case of $(\tilde{\Theta}_c,\tilde{\Theta}_s)=(\frac{\pi}{2},0)$.

(2) Case of $R_c=0$

From eq. (44), $\tilde{R}=|R_s\sin\tilde{\Theta}_s|$ follows. Therefore, for $\tilde{\Theta}_s=(0,\pm\frac{\pi}{2},\pi)$, $\tilde{R}$ becomes $(0,R_s,0)$, respectively. Thus, we obtain $\tilde{\Theta}_s=\pm\frac{\pi}{2}$, and from eq. (43), $R_s\sin\overline{\theta}=R_s\sin\tilde{\Theta}_s=\pm R_s$ follows. Thus, we obtain $\overline{\theta}=\pm\frac{\pi}{2}$. Since we have

$$\tilde{\Xi}(\theta) = \sqrt{(J_0R)^2+(J_1R_s\sin\theta)^2}, \quad (A\cdot 17)$$

the present solution coincides with the solution obtained by putting $R_c=0$ in solutions 2 and 3.

(3) Case of $R_s=0$

From eq. (44), $\tilde{R}=|R_c\sin\tilde{\Theta}_c|$ follows. Therefore, for $\tilde{\Theta}_c=(0,\pm\frac{\pi}{2},\pi)$, $\tilde{R}$ becomes $(0,R_c,0)$, respectively. Thus, we obtain $\tilde{\Theta}_c=\pm\frac{\pi}{2}$, and from eq. (43), $R_c\cos\overline{\theta}=R_c\sin\tilde{\Theta}_c=\pm R_c$ follows.

Thus, we obtain $\overline{\theta} = 0, \pi$. Since we have

$$\tilde{\Xi}(\theta) \quad = \quad \sqrt{(J_0 R)^2 + (J_1 R_c \sin\theta)^2}, \tag{A·18}$$

the present solution coincides with the solution obtained by putting $R_s = 0$ in solutions 4 and 5.

Therefore, for $\tilde{R} \neq 0$, solutions other than solution 1 are

$$(\tilde{\Theta}_c, \tilde{\Theta}_s) = (0, \frac{\pi}{2}),\ \tilde{R} = R_s,\ \overline{\theta} = \frac{\pi}{2}, \tag{A·19}$$

$$(\tilde{\Theta}_c, \tilde{\Theta}_s) = (0, -\frac{\pi}{2}),\ \tilde{R} = R_s,\ \overline{\theta} = -\frac{\pi}{2}, \tag{A·20}$$

$$\tilde{\Xi}(\theta) = \sqrt{(J_0 R - J_1 R_c \cos\theta)^2 + (J_1 R_s \sin\theta)^2} \tag{A·21}$$

and the solution with $R_c = 0$ in this solution, and the solution

$$(\tilde{\Theta}_c, \tilde{\Theta}_s) = (\frac{\pi}{2}, 0),\ \overline{\theta} = 0 \tag{A·22}$$

$$(\tilde{\Theta}_c, \tilde{\Theta}_s) = (-\frac{\pi}{2}, 0),\ \overline{\theta} = \pi \tag{A·23}$$

$$\tilde{\Xi}(\theta) = \sqrt{(J_0 R - J_1 R_s \cos\theta)^2 + (J_1 R_c \sin\theta)^2}, \tag{A·24}$$

and the solution with $R_s = 0$ in this solution.

A.2 Case of $\tilde{R} = 0$

The conditions are

$$R_c \sin\tilde{\Theta}_c = 0 \text{ and } R_s \sin\tilde{\Theta}_s = 0.$$

Therefore, from eq. (13), we obtain

$$\Xi(\theta) \quad = \quad |J_0 R + J_1 (R_c \cos\tilde{\Theta}_c \cos\theta + R_s \cos\tilde{\Theta}_s \sin\theta)|. \tag{A·25}$$

From the conditions, we have

$$\begin{aligned} &\{R_c = 0,\ \text{or}\ \tilde{\Theta}_c = 0,\ \text{or}\ \tilde{\Theta}_c = \pi\} \\ &\text{and} \\ &\{R_s = 0,\ \text{or}\ \tilde{\Theta}_s = 0,\ \text{or}\ \tilde{\Theta}_s = \pi\}. \end{aligned}$$

Therefore, we obtain the following cases:

(1) $R_c = 0, R_s = 0 \to R_1 = 0 \to$ Kuramoto model

(2) $R_c = 0, \tilde{\Theta}_s = 0, \to \Xi(\theta) = |J_0 R + J_1 R_1 \sin\theta|,\ (R_1 = R_s)$,

(3) $R_c = 0, \tilde{\Theta}_s = \pi, \to \Xi(\theta) = |J_0 R - J_1 R_1 \sin\theta|,\ (R_1 = R_s)$,

(4) $R_s = 0, \tilde{\Theta}_c = 0, \to, \Xi(\theta) = |J_0 R + J_1 R_1 \cos\theta|,\ (R_1 = R_c)$,

(5) $R_s = 0, \tilde{\Theta}_c = \pi, \to \Xi(\theta) = |J_0 R - J_1 R_1 \cos\theta|,\ (R_1 = R_c)$,

(6) $\tilde{\Theta}_c = 0, \tilde{\Theta}_s = 0, \rightarrow \Xi(\theta) = |J_0 R + J_1(R_c \cos\theta + R_s \sin\theta)|$,
(7) $\tilde{\Theta}_c = 0, \tilde{\Theta}_s = \pi, \rightarrow \Xi(\theta) = |J_0 R + J_1(R_c \cos\theta - R_s \sin\theta)|$,
(8) $\tilde{\Theta}_c = \pi, \tilde{\Theta}_s = 0, \rightarrow \Xi(\theta) = |J_0 R - J_1(R_c \cos\theta - R_s \sin\theta)|$,
(9) $\tilde{\Theta}_c = \pi, \tilde{\Theta}_s = \pi, \rightarrow \Xi(\theta) = |J_0 R - J_1(R_c \cos\theta + R_s \sin\theta)|$.

f is given by

$$= \frac{1}{2}(J_0 R^2 + J_1 R_1^2) - \frac{1}{\beta}\frac{1}{2\pi}\int_0^{2\pi} d\theta \ln\{2\pi I_0(\beta\Xi(\theta))\}. \quad \text{(A·26)}$$

By the coordinate transformations, such as $\theta = \theta' + \pi, \theta = \frac{\pi}{2} - \theta'$, and $\theta' = \theta \pm \hat{\theta}$, and their combinations, cases (3) to (9) become the same as case (2). Here, we define $R_c = R_1 \cos\hat{\theta}$ and $R_s = R_1 \sin\hat{\theta}$ for cases (6) to (9). Furthermore, case 2 coincides with solutions 2 and 3 of $\tilde{R} \neq 0$ with $R_s = 0$, and it also coincides with the solutions 4 and 5 of $\tilde{R} \neq 0$ when $R_c = 0$. Therefore, solutions for the case of $\tilde{R} = 0$ are derived from the solutions for the case of $\tilde{R} \neq 0$.

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