Dynamical Hehaviour of Phase Oscillator Networks on the Bethe Lattice

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We study the dynamical nature of phase oscillator networks on a Bethe lattice. We derive self-consistent equations for the cavity fields (i.e. the oscillator probability distributions) by using the cavity method both in equilibrium and via an analytic approximation for non-equilibrium states. The order parameters and their evolution equations are obtained as functions of the cavity fields. The theoretical results and the accuracy of our approximations are confirmed by comparison with simulations involving direct integration of the Langevin equations describing the microscopic oscillator dynamics.

KEYWORDS: oscillator network, finite connectivity, bethe lattice

1. Introduction

We are interested in the dynamics of systems where elements have their own dynamics and each individual element interacts with only a finite number of other elements. One such example is a dynamical system model of an immune network,[1,2] which we have been studying in recent years.[3] However, this model has proved rather complicated to study analytically, even using approximate dynamical techniques. It is therefore desirable to study models in which analytical treatment is possible but which retain sufficiently complex behaviour to be of some interest.

In recent years, analytical approaches to investigate such models have been developed,[4] and the equilibrium states have been studied. In this paper, we study phase oscillator networks on the Bethe lattice as an example of a fully solvable equilibrium model where we can develop analytic approximation techniques to investigate the dynamics, which have been studied previously using an approximation based on the path integral approach.[5] Previously, the planar rotator model on the Bethe lattice, which is nothing but the phase oscillator networks with constant natural frequency, has been studied in the equilibrium and the phase transition temperature and the magnetization have been derived.[6] Further, in more general chiral models the equilibrium states have been studied by using the replica theory[4] while in more general dilute systems the cavity method has been applied to equilibrium problems.[7] Using the cavity method[7,8] we calculate self-consistent equations of the cavity field both in

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equilibrium and non-equilibrium states. The non-equilibrium analysis provides the novel features of this work although the equilibrium analysis is instructive in developing intuition and the mathematical technology in a more familiar setting. In particular, the equilibrium order parameters and their dynamic evolution equations can both be obtained by using cavity fields. In the remainder of this section we describe the model we will study in more detail.

We study N coupled phase oscillators, as introduced by Kuramoto.[9,10] Let ϕ_i be the phase of the i-th oscillator. The evolution equation for ϕ_i is given by

$$\frac{d}{dt}\phi_i = \omega_i + \sum_{j\neq i} J_{ij}\sin(\phi_j - \phi_i) + \eta_i, \tag{1}$$

where $\eta_i(t)$ is a Gaussian white noise process with variance $2T$,

$$\left\langle \eta_i(t)\eta_j(t')\right\rangle = 2T\delta_{ij}\delta(t-t'). \tag{2}$$

In this paper, to make the problem tractable, we set $\omega_i = \omega = 0$ for all i. Further, taking $J_{ij} = J_{ji}$, we obtain the following evolution equation,

$$\frac{d}{dt}\phi_i = -\frac{\partial}{\partial\phi_i}H + \eta_i, \tag{3}$$

$$H(\boldsymbol{\phi}) = -\sum_{i<j} J_{ij}\cos(\phi_i - \phi_j). \tag{4}$$

Now, let us consider the oscillators on the Bethe lattice given by $\{J_{ij}\}$ satisfying the following constraints:

$$\sum_{j(\neq i)} J_{ij} = cJ \text{ for any } i, \quad J_{ij} = J_{ji}, \quad J_{ij} \in \{0, J\}. \tag{5}$$

Each oscillator interacts with only c other oscillators and it is assumed that there is no finite loops in the infinite system (or rather that there are only such a small number as to be thermodynamically insignificant, e.g. $\mathcal{O}(N^0)$). The number of connections c of each oscillator is of order N^0, that is, it is a constant independent of the system size N. In the below, we take c_{ij} to be 1 if there is a connection between i-th and j-th oscillators and 0 otherwise. The dynamical behaviour of this model may also be studied by dynamical replica theory.[11] However, it is rather difficult to solve the evolution equations by this method. In this paper using the cavity method,[7] we solve those equations for a simple, finite set of order parameters.

In section 2, we study equilibrium states and then we derive the evolution equations of observables in section 3. Theoretical results and numerical results, which are obtained by direct integration of the Langevin equations (1), are compared in section 4 before we conclude in section 5.

2. Equilibrium States

The probability density of $\boldsymbol{\phi} = (\phi_1, \cdots, \phi_N)$ in equilibrium, $p_{eq}(\boldsymbol{\phi})$, can be expressed as

$$p_{eq}(\boldsymbol{\phi}) = \frac{1}{Z}e^{-\beta H}, \tag{6}$$

$$Z = \int_0^{2\pi} d\boldsymbol{\phi} e^{-\beta H}, \tag{7}$$

where $\int_0^{2\pi} d\boldsymbol{\phi}$ implies an N-dimensional integration and $\beta = 1/T$. We define the following order parameters:

$$m_c = \frac{1}{N}\sum_i \cos(\phi), \tag{8}$$

$$m_s = \frac{1}{N}\sum_i \sin(\phi), \tag{9}$$

$$m = \sqrt{m_c^2 + m_s^2}. \tag{10}$$

Let us introduce the cavity field $P_i^{(j)}(\phi_i)$. $P_i^{(j)}(\phi_i)$ is the probability density of the phase of the i-th oscillator when one of its neighbouring oscillators, the j-th oscillator, is removed from the system. Since every point on the Bethe lattice is equivalent, we assume $P_i^{(j)}(\phi_i)$ is independent of the lattice point i and the neighbouring point j, and we denote it by $P^{\rm cav}(\phi_i)$. Next, we can write a self-consistent functional equation for $P^{\rm cav}(\phi)$ as

$$P^{\rm cav}(\phi) = \frac{1}{Z_{\rm cav}}\{\int_0^{2\pi} d\phi' e^{\beta J\cos(\phi-\phi')}P^{\rm cav}(\phi')\}^{c-1}, \tag{11}$$

where $Z_{\rm cav}$ is just the normalization constant for this probability distribution. The probability density of the phase of the i-th oscillator, i.e. the density of the marginal distribution, is denoted by $P^{\rm true}(\phi_i)$ and may be calculated as

$$P^{\rm true}(\phi) = \frac{1}{Z_{\rm true}}\{\int_0^{2\pi} d\phi' e^{\beta J\cos(\phi-\phi')}P^{\rm cav}(\phi')\}^{c}, \tag{12}$$

where $Z_{\rm true}$ is again just a normalization constant. Once the marginal distribution for the spins $P^{\rm true}(\phi_i)$ has been obtained, the order parameters can be expressed rather intuitively as:

$$m_c = \int_0^{2\pi} d\phi\ \cos(\phi)P^{\rm true}(\phi), \tag{13}$$

$$m_s = \int_0^{2\pi} d\phi\ \sin(\phi)P^{\rm true}(\phi). \tag{14}$$

This system exhibits two phases, the ordered (Ferromagnetic) phase of $m > 0$ (F) and the disordered (Paramagnetic) phase $m = 0$ (P). The transition temperature between the two phases is determined by

$$c - 1 = \frac{I_0(\beta J)}{I_1(\beta J)}, \tag{15}$$

where I_n are the modified Bessel functions.

$$I_n(z) \equiv \int_0^{2\pi} \frac{d\phi}{2\pi}\cos(n\phi)e^{z\cos(\phi)}. \tag{16}$$

Now, let us derive eq.(15). At high temperatures, $m_c = m_s = 0$ holds because the thermal noise dominates and the disordered state, that is the Paramagnetic phase, appears. In this case, we expect $P^{\rm cav}(\phi) = \frac{1}{2\pi}$. This is indeed a solution of eq.(11) for any temperature T. Now, let us put

$$P^{\rm cav}(\phi) = \frac{1}{2\pi} + \Delta(\phi), \tag{17}$$

where $|\Delta(\phi)| \ll 1$ for any ϕ. Inserting eq.(17) into eq.(11) and keeping all terms up to the first order in $\Delta(\phi)$, we obtain

$$Z_{\rm cav} = \frac{1}{2\pi}\{I_0(\beta J)\}^{c-1}, \tag{18}$$

$$\Delta(\phi) = \frac{c-1}{I_0(\beta J)}\int_0^{2\pi}\frac{d\phi'}{2\pi}e^{\beta J\cos(\phi-\phi')}\Delta(\phi'). \tag{19}$$

Let us expand $\Delta(\phi)$ in a Fourier series.

$$\Delta(\phi) = \sum_{n=1}^{\infty}\{a_n\cos(n\phi) + b_n\sin(n\phi)\}. \tag{20}$$

The constant term $a_0 = 0$ is 0, because $\int_0^{2\pi}\Delta(\phi)d\phi = 0$ which is enforced by normalisation of $P^{\rm cav}(\phi)$. Inserting this definition into eq.(19), we obtain the self-consistent equations

$$a_n = \frac{c-1}{I_0(\beta J)}a_n I_n(\beta J), \tag{21}$$

$$b_n = \frac{c-1}{I_0(\beta J)}b_n I_n(\beta J). \tag{22}$$

Thus, if $c - 1 = \frac{I_0(\beta J)}{I_l(\beta J)}$, then a_l and b_l are arbitrary and other as and bs are 0 (as this is the only possible solution to (22)). Since $I_n(x) > 0$, and $I_n(x) < I_1(x)$ for $n > 1$ and $x > 0$, the transition temperature between the Ferromagnetic and Paramagnetic phases is determined by eq.(15).

Now, let us explain one approach to the numerical evaluation of these equations. We are able to solve eq.(11) in a straightforward manner by using the iteration method:

$$P^{\rm cav}_{t+1}(\phi) = \frac{1}{Z_{{\rm cav},t}}\{\int_0^{2\pi}d\phi' e^{\beta J\cos(\phi'-\phi)}P^{\rm cav}_t(\phi')\}^{c-1}. \tag{23}$$

We tried various functions for the initial distribution $P^{\rm cav}_0(\phi)$ and found that if the resultant function is redefined after convergence so that it attains its maximum at $\phi = \pi$, it is a unique unimodal distribution. Then, we found that redefined functions $P^{\rm cav}(\phi)$ and $P^{\rm true}(\phi)$ almost satisfy the symmetry $P(\pi+\phi) = P(\pi-\phi)$ (at least within the accuracy that the numerical integrations can be performed - and certainly a symmetric solution is indeed a solution of the update equations). Thus, we obtain $m_s \simeq 0$ and $m \simeq |m_c|$.

As a comparison, we solved the Langevin equation (3) by the Euler method. We took a time increment of $\Delta t = 0.01$, and about 10 realisations of the dynamics. We performed simulations for several values of N, $N = 1,000,\ 2,000,\ 4,000$ and 6,000, and obtained similar

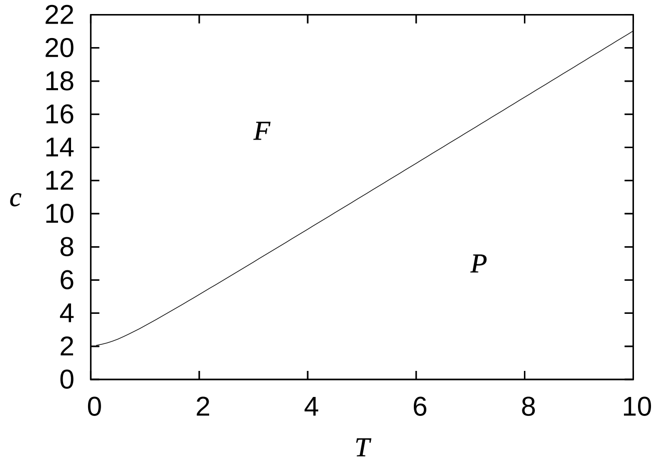

Fig. 1. Phase diagram in $T - c$ plane.

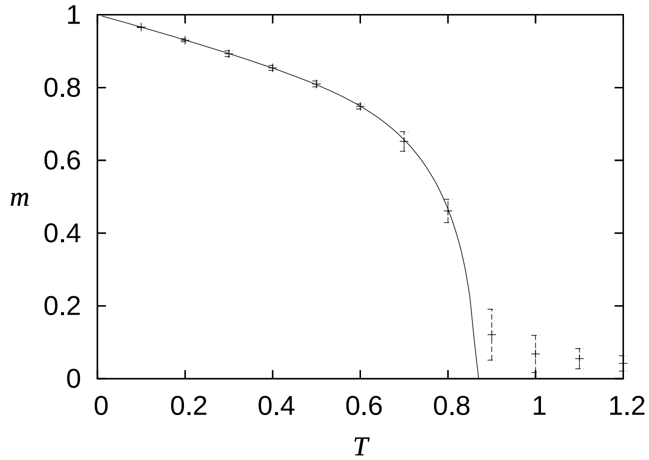

Fig. 2. Temperature dependence of m. $c = 3$. Solid curve: theory, symbols with errorbars: simulation of the langevin eq. $N = 2,000$, average of 10 samples.

results. In Fig.1, we display the phase diagram. The temperature dependence of m is displayed together with results of numerical simulations in Fig.2. The agreement between theoretical results and numerical simulations is quite acceptable. Above the critical temperature, in our simulations, m has non-zero values. One reason for this is that the value of m is non-negative by definition. Thus, even in the paramagnetic phase, the value of m is non-zero due to finite size fluctuation effects. In the next section, we study the dynamical behaviour of the model.

3. Dynamical Behaviour

The probability density of $\boldsymbol{\phi}$ at time t, $p_t(\boldsymbol{\phi})$, obeys the following Fokker-Planck equation,[14]

$$\frac{\partial}{\partial t} p_t(\boldsymbol{\phi}) = \sum_i \frac{\partial}{\partial \phi_i} \{ \frac{\partial H}{\partial \phi_i} p_t(\boldsymbol{\phi}) \} + T \sum_i \frac{\partial^2}{\partial \phi_i^2} p_t(\boldsymbol{\phi}). \tag{24}$$

We introduce a set of intensive macroscopic observables $\mathbf{\Omega}(\boldsymbol{\phi}) = (\Omega_1(\boldsymbol{\phi}), \cdots, \Omega_k(\boldsymbol{\phi}))$ and define their probability density $p_t(\mathbf{\Omega})$ as

$$p_t(\mathbf{\Omega}) = \int_0^{2\pi} d\boldsymbol{\phi} p_t(\boldsymbol{\phi}) \delta(\mathbf{\Omega} - \mathbf{\Omega}(\boldsymbol{\phi})). \tag{25}$$

From this, we can derive a Fokker-Planck equation by the standard recipe.

$$\begin{aligned}
\frac{\partial}{\partial t} p_t(\mathbf{\Omega}) &= \int_0^{2\pi} d\boldsymbol{\phi} \frac{\partial}{\partial t} p_t(\boldsymbol{\phi}) \delta(\mathbf{\Omega} - \mathbf{\Omega}(\boldsymbol{\phi})) \\
&= \int_0^{2\pi} d\boldsymbol{\phi} [\sum_i \frac{\partial}{\partial \phi_i} \{\frac{\partial H}{\partial \phi_i} p_t(\boldsymbol{\phi})\} + T \sum_i \frac{\partial^2}{\partial \phi_i^2} p_t(\boldsymbol{\phi})] \delta(\mathbf{\Omega} - \mathbf{\Omega}(\boldsymbol{\phi})) \\
&= \int_0^{2\pi} d\boldsymbol{\phi} \left[- \sum_i \{\frac{\partial H}{\partial \phi_i} p_t(\boldsymbol{\phi})\} \frac{\partial}{\partial \phi_i} + T \sum_i p_t(\boldsymbol{\phi}) \frac{\partial^2}{\partial \phi_i^2} \right] \delta(\mathbf{\Omega} - \mathbf{\Omega}(\boldsymbol{\phi})) \\
&= \int_0^{2\pi} d\boldsymbol{\phi} p_t(\boldsymbol{\phi}) \\
&\quad \times \left[\sum_i \sum_\mu \left(\frac{\partial H}{\partial \phi_i} \frac{\partial \Omega_\mu}{\partial \phi_i} - T \frac{\partial^2 \Omega_\mu}{\partial \phi_i^2} \right) \frac{\partial}{\partial \Omega_\mu} + T \sum_i \sum_{\mu\nu} \frac{\partial \Omega_\mu}{\partial \phi_i} \frac{\partial \Omega_\nu}{\partial \phi_i} \frac{\partial^2}{\partial \Omega_\mu \partial \Omega_\nu} \right] \delta(\mathbf{\Omega} - \mathbf{\Omega}(\boldsymbol{\phi})) \\
&= - \sum_\mu \frac{\partial}{\partial \Omega_\mu} \left[p_t(\mathbf{\Omega}) \langle \sum_i \left(- \frac{\partial H}{\partial \phi_i} \frac{\partial \Omega_\mu}{\partial \phi_i} + T \frac{\partial^2 \Omega_\mu}{\partial \phi_i^2} \right) \rangle_{\mathbf{\Omega}} \right] \\
&\quad + T \sum_{\mu\nu} \frac{\partial}{\partial \Omega_\mu} \frac{\partial}{\partial \Omega_\nu} \sum_i \left[p_t(\mathbf{\Omega}) \langle \frac{\partial \Omega_\mu}{\partial \phi_i} \frac{\partial \Omega_\nu}{\partial \phi_i} \rangle_{\mathbf{\Omega}} \right],
\end{aligned} \tag{26}$$

where the $\langle \ldots \rangle_{\mathbf{\Omega}}$ denotes averaging over the microscopic probability measure $p_t(\boldsymbol{\phi})$ for those states in which the macroscopic observables $\mathbf{\Omega}(\boldsymbol{\phi}) = \mathbf{\Omega}$. We call the subspace $\boldsymbol{\phi} \in [0, 2\pi]^N$ where $\mathbf{\Omega}(\boldsymbol{\phi}) = \mathbf{\Omega}$, the subshell, and call the average over the subshell the subshell average. It can be shown that for the choices we take for $\mathbf{\Omega}(\boldsymbol{\phi})$, the diffusion terms in equation (26) disappear and we are left merely with a Liouville equation.

$$\frac{\mathrm{d}}{\mathrm{d}t} \Omega_\mu(t) = \langle \sum_i \left[\sum_j c_{ij} \sin(\phi_j - \phi_i) + T \frac{\partial}{\partial \phi_i} \right] \frac{\partial}{\partial \phi_i} \Omega_\mu(\boldsymbol{\phi}) \rangle_{\mathbf{\Omega}}. \tag{27}$$

We have taken $J = 1$ for simplicity. Now, we make the maximum entropy assumption that $p_t(\boldsymbol{\phi})$ is uniform in the subshell, i.e. all $\boldsymbol{\phi}$ such that $\mathbf{\Omega}(\boldsymbol{\phi}) = \mathbf{\Omega}$ are equally likely. Hence, the average in eq. (27) simplifies under this assumption to:

$$\langle \ldots \rangle_{\mathbf{\Omega}} = \frac{\int_0^{2\pi} \mathrm{d}\boldsymbol{\phi} \delta[\mathbf{\Omega} - \mathbf{\Omega}(\boldsymbol{\phi})](\ldots)}{\int_0^{2\pi} \mathrm{d}\boldsymbol{\phi} \delta[\mathbf{\Omega} - \mathbf{\Omega}(\boldsymbol{\phi})]}. \tag{28}$$

3.1 3 Order Parameter Scheme (3OPS)

To begin with we consider the simplest set of three observables that could be hoped to describe the system:

$$m_c(\boldsymbol{\phi}) = \frac{1}{N} \sum_i \cos(\phi_i), \tag{29}$$

$$m_s(\boldsymbol{\phi}) = \frac{1}{N}\sum_i \sin(\phi_i), \tag{30}$$

$$e(\boldsymbol{\phi}) = \frac{1}{N}\sum_{i<j} c_{ij}\cos(\phi_i - \phi_j). \tag{31}$$

We call this scheme 3OPS. The energy is an obvious choice, while m_c and m_s allow the description of overall ordering among the oscillators. Inserting these observables into the equation (27) in turn gives us coupled ordinary differential equations (ODEs) which describe the system's behaviour:

$$\frac{\mathrm{d}}{\mathrm{d}t}m_c(t) = -\frac{1}{N}\sum_i \langle \sum_j c_{ij}\sin(\phi_j - \phi_i)\sin(\phi_i)\rangle_{m_c(t),m_s(t),e(t)} - Tm_c(t), \tag{32}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}m_s(t) = \frac{1}{N}\sum_i \langle \sum_j c_{ij}\sin(\phi_j - \phi_i)\cos(\phi_i)\rangle_{m_c(t),m_s(t),e(t)} - Tm_s(t), \tag{33}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}e(t) = \frac{1}{N}\sum_i \langle \left(\sum_j c_{ij}\sin(\phi_j - \phi_i)\right)^2 \rangle_{m_c(t),m_s(t),e(t)} - 2Te(t). \tag{34}$$

The non-trivial aspect of these coupled ODEs is the measure. We have to average over all states $\boldsymbol{\phi}$ according to definition (28). To do this we move to the canonical framework, writing:

$$\delta[\boldsymbol{\Omega} - \boldsymbol{\Omega}(\boldsymbol{\phi})] = \exp[\hat{e}\sum_{i<j} c_{ij}\cos(\phi_i - \phi_j) + \hat{m}_c\sum_i \cos(\phi_i) + \hat{m}_s\sum_i \sin(\phi_i)]. \tag{35}$$

The problem is reduced to finding the triple $\{\hat{e}, \hat{m}_c, \hat{m}_s\}$ at each time t for which the canonical measure satisfies the equalities:

$$m_c = \langle m_c(\boldsymbol{\phi})\rangle_{m_c,m_s,e}, \tag{36}$$

$$m_s = \langle m_s(\boldsymbol{\phi})\rangle_{m_c,m_s,e}, \tag{37}$$

$$e = \langle e(\boldsymbol{\phi})\rangle_{m_c,m_s,e}. \tag{38}$$

This is a highly non-linear three dimensional inverse problem and computationally this is the most challenging part for finitely connected random systems, since this inverse problem must be solved for each time t.

This method is applicable to any finite connectivity system, although solving the equations numerically is very hard task. Fortunately, we can take advantage of the Bethe lattice and make the problem simpler. To do so, we introduce the cavity field $P^{\mathrm{cav}}(\phi)$ and the marginal distribution $P^{\mathrm{true}}(\phi)$ as in the equilibrium case. Then, we obtain the functional equation for $P^{\mathrm{cav}}(\phi)$ as

$$P^{\mathrm{cav}}(\phi) = \frac{1}{Z_{\mathrm{cav}}} e^{\hat{m}_c\cos(\phi)+\hat{m}_s\sin(\phi)}\{\int_0^{2\pi} d\phi' e^{\hat{e}\cos(\phi-\phi')}P^{\mathrm{cav}}(\phi')\}^{c-1}. \tag{39}$$

$P^{\rm true}(\phi)$ is calculated as

$$P^{\rm true}(\phi) = \frac{1}{Z_{\rm true}} e^{\hat{m}_c \cos(\phi)+\hat{m}_s \sin(\phi)} \{ \int_0^{2\pi} d\phi' e^{\hat{e}\cos(\phi-\phi')} P^{\rm cav}(\phi') \}^c, \tag{40}$$

Observables are expressed as

$$m_c = \int_0^{2\pi} d\phi \ \cos(\phi) P^{\rm true}(\phi), \tag{41}$$

$$m_s = \int_0^{2\pi} d\phi \ \sin(\phi) P^{\rm true}(\phi), \tag{42}$$

$$e = \frac{c}{2} \langle \cos(\phi-\phi') \rangle_2^{\rm 3OPS}, \tag{43}$$

where $\langle f(\phi,\phi') \rangle_2^{\rm 3OPS}$ is defined as

$$\langle f(\phi,\phi') \rangle_2^{\rm 3OPS} \equiv \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' P_2^{\rm 3OPS}(\phi,\phi') f(\phi,\phi'), \tag{44}$$

$$P_2^{\rm 3OPS}(\phi,\phi') \equiv \frac{1}{Z_2} P^{\rm cav}(\phi) P^{\rm cav}(\phi') e^{\hat{e}\cos(\phi-\phi')}. \tag{45}$$

where Z_2 is a normalization constant and is equal to $\frac{Z_{\rm true}}{Z_{\rm cav}}$. Thus, we obtain the self-consistent equations, (39-43). Then we obtain the following ODEs for observables:

$$\frac{\rm d}{{\rm d}t} m_c(t) = -c \langle \sin(\phi-\phi') \sin(\phi') \rangle_2^{\rm 3OPS} - T m_c(t), \tag{46}$$

$$\frac{\rm d}{{\rm d}t} m_s(t) = c \langle \sin(\phi-\phi') \cos(\phi') \rangle_2^{\rm 3OPS} - T m_s(t), \tag{47}$$

$$\frac{\rm d}{{\rm d}t} e(t) = \frac{\int_0^{2\pi} d\phi \{\prod_{l=1}^c \int_0^{2\pi} d\phi_l P^{\rm cav}(\phi_l)\} e^{\hat{e}\sum_l \cos(\phi-\phi_l)+\hat{m}_c\cos(\phi)+\hat{m}_s\sin(\phi)} (\sum_l \sin(\phi_l-\phi))^2}{\int_0^{2\pi} d\phi \{\prod_{l=1}^c \int_0^{2\pi} d\phi_l P^{\rm cav}(\phi_l)\} e^{\hat{e}\sum_l \cos(\phi-\phi_l)+\hat{m}_c\cos(\phi)+\hat{m}_s\sin(\phi)}} - 2Te(t). \tag{48}$$

The last equation can be further rewritten as

$$\frac{\rm d}{{\rm d}t} e(t) = c(c-1) \langle \sin(\phi'-\phi) \sin(\phi''-\phi) \rangle_3^{\rm 3OPS} + c \langle \sin^2(\phi'-\phi) \rangle_2^{\rm 3OPS} - 2Te(t), \tag{49}$$

where $\langle f(\phi,\phi',\phi'') \rangle_3^{\rm 3OPS}$ is defined as

$$\langle f(\phi,\phi',\phi'') \rangle_3^{\rm 3OPS} \equiv \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' \int_0^{2\pi} d\phi'' P_3^{\rm 3OPS}(\phi,\phi',\phi'') f(\phi,\phi',\phi''), \tag{50}$$

$$P_3^{\rm 3OPS}(\phi,\phi',\phi'') \equiv \frac{1}{Z_3} P^{\rm cav}(\phi') P^{\rm cav}(\phi'') e^{\hat{e}\{\cos(\phi'-\phi)+\cos(\phi''-\phi)\}+\hat{m}_c\cos(\phi)+\hat{m}_s\sin(\phi)} \times \{ \int_0^{2\pi} d\psi P^{\rm cav}(\psi) e^{\hat{e}\cos(\psi-\phi)} \}^{c-2}, \tag{51}$$

where Z_3 is a normalization constant and is equal to $Z_{\rm true}$. In this scheme, we only have to perform up to 3-dimensional integration to solve the self-consistent equations for $\hat{m}_c, \hat{m}_s$ and $\hat{e}$ in order to estimate the right-hand sides of the evolution equations.

When m_c, m_s and e are given, $\hat{m}_c, \hat{m}_s, \hat{e}$ and $P^{\rm cav}(\phi)$ are determined by solving the self-consistent equations (39-43) which is essentially a three dimensional inverse problem. By using these values, we can estimate the derivatives of m_c, m_s and e at time t with respect to t from (46-48). In this way, we can integrate ODEs (46-48) by using the Euler method. Here, we should note the following. In the present scheme, we cannot treat the case in which an initial condition $\phi_i(0)$ is not drawn from the probability distribution $P^{\rm true}(\phi)$, because we assume that $p_t(\boldsymbol{\phi})$ is uniform in any subshell. Further, we have implicitly assumed site equivalence (i.e. the marginal distribution of $P^{cav}(\phi)$ is identical on all cavity sites) which means that this approach in its current form will not work on a specific instance of a graph. This is not a theoretical barrier but if site symmetry is not assumed then numerically the complexity of the task increases by a factor Nc, as we require a cavity distribution for each cavity site. This problem would require significant computational resources, or algorithmic improvements, to tackle.

Further, we can prove that the equilibrium solution is a stationary states of the ODEs. See Appendix.

3.2 4 Order Parameter Scheme (4OPS)

Next, we add another observable and call the scheme 4OPS. The observable we add is

$$m_{ss}(\boldsymbol{\phi}) = \frac{1}{N}\sum_{ij} c_{ij}\sin(\phi_i)\sin(\phi_j - \phi_i). \tag{52}$$

The rationale for adding this variable is that given $m_{ss}(\boldsymbol{\phi})$ the equation for $m_c(\boldsymbol{\phi})$ is closed in terms of the available variables. In theory one can keep playing the same game adding variables with longer range interactions which will more precisely specify the subshell. In practice, the numerical solution of the equations becomes rapidly more CPU intensive. We assume the following canonical distribution for $\delta[\boldsymbol{\Omega} - \boldsymbol{\Omega}(\boldsymbol{\phi})]$ as in 3OPS case.

$$\begin{aligned}\delta[\boldsymbol{\Omega} - \boldsymbol{\Omega}(\boldsymbol{\phi})] \quad &= \\ \exp[\hat{e}\sum_{i<j} c_{ij}\cos(\phi_i - \phi_j) + \hat{m}_c\sum_i \cos(\phi_i) &+ \hat{m}_s\sum_i \sin(\phi_i) + \hat{m}_{ss}\sum_{ij} c_{ij}\sin(\phi_i)\sin(\phi_j - \phi_i)].\end{aligned} \tag{53}$$

Then, we obtain the functional equation for $P^{\rm cav}(\phi)$ as

$$\begin{aligned}P^{\rm cav}(\phi) \quad &= \quad \frac{1}{Z_{\rm cav}} e^{\hat{m}_c\cos(\phi) + \hat{m}_s\sin(\phi)} \\ &\quad \times\{\int_0^{2\pi} d\phi' e^{\hat{e}\cos(\phi-\phi') + \hat{m}_{ss}\{\sin(\phi)-\sin(\phi')\}\sin(\phi'-\phi)} P^{\rm cav}(\phi')\}^{c-1}.\end{aligned} \tag{54}$$

$P^{\rm true}(\phi)$ is calculated as

$$\begin{aligned}P^{\rm true}(\phi) \quad &= \quad \frac{1}{Z_{\rm true}} e^{\hat{m}_c\cos(\phi) + \hat{m}_s\sin(\phi)} \\ &\quad \times\{\int_0^{2\pi} d\phi' e^{\hat{e}\cos(\phi-\phi') + \hat{m}_{ss}\{\sin(\phi)-\sin(\phi')\}\sin(\phi'-\phi)} P^{\rm cav}(\phi')\}^{c}.\end{aligned} \tag{55}$$

Expressions for m_c and m_s are same as before. e and m_{ss} are expressed as

$$e = \frac{c}{2}\langle \cos(\phi'-\phi)\rangle_2^{\rm 4OPS}, \tag{56}$$

$$m_{ss} = c\langle \sin(\phi)\sin(\phi'-\phi)\rangle_2^{\rm 4OPS}, \tag{57}$$

where $\langle f(\phi,\phi')\rangle_2^{\rm 4OPS}$ is defined as

$$\langle f(\phi,\phi')\rangle_2^{\rm 4OPS} \equiv \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' P_2^{\rm 4OPS}(\phi,\phi') f(\phi,\phi'), \tag{58}$$

$$P_2^{\rm 4OPS}(\phi,\phi') \equiv \frac{1}{Z_2} P^{\rm cav}(\phi) P^{\rm cav}(\phi') e^{\hat{e}\cos(\phi-\phi') + \hat{m}_{ss}\{\sin(\phi)-\sin(\phi')\}\sin(\phi'-\phi)}. \tag{59}$$

Finally, we obtain the ODEs for observables as

$$\frac{\mathrm{d}}{\mathrm{d}t} m_c(t) = -c\langle \sin(\phi-\phi')\sin(\phi')\rangle_2^{\rm 4OPS} - T m_c(t), \tag{60}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} m_s(t) = c\langle \sin(\phi-\phi')\cos(\phi')\rangle_2^{\rm 4OPS} - T m_s(t), \tag{61}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} e(t) = c(c-1)\langle \sin(\phi'-\phi)\sin(\phi''-\phi)\rangle_3^{\rm 4OPS} + c\langle \sin^2(\phi'-\phi)\rangle_2^{\rm 4OPS} - 2Te(t), \tag{62}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} m_{ss}(t) = {}& c(c-1)\langle \sin(\phi'-\phi)\{\cos(\phi)\sin(\phi''-\phi) + (\sin\phi'' - \sin\phi)\cos(\phi''-\phi)\}\rangle_3^{\rm 4OPS} \\ & + c\langle \sin(\phi'-\phi)\{\cos(\phi)\sin(\phi'-\phi) - 2\sin\phi\cos(\phi'-\phi)\}\rangle_2^{\rm 4OPS} \\ & - 2Tc\langle \cos(\phi'-\phi)\cos\phi'\rangle_2^{\rm 4OPS} - 3T m_{ss}(t),\end{aligned} \tag{63}$$

where $\langle f(\phi,\phi',\phi'')\rangle_3^{\rm 4OPS}$ is defined as

$$\langle f(\phi,\phi',\phi'')\rangle_3^{\rm 4OPS} \equiv \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' \int_0^{2\pi} d\phi'' P_3^{\rm 4OPS}(\phi,\phi',\phi'') f(\phi,\phi',\phi''), \tag{64}$$

$$\begin{aligned}P_3^{\rm 4OPS}(\phi,\phi',\phi'') \equiv {}& \frac{1}{Z_3} P^{\rm cav}(\phi') P^{\rm cav}(\phi'') \\ & \times e^{\hat{e}\{\cos(\phi'-\phi)+\cos(\phi''-\phi)\} + \hat{m}_{ss}\{\sin(\phi)-\sin(\phi')\}\sin(\phi'-\phi) + \hat{m}_{ss}\{\sin(\phi)-\sin(\phi'')\}\sin(\phi''-\phi)} \\ & \times e^{\hat{m}_c\cos(\phi)+\hat{m}_s\sin(\phi)} \{\int_0^{2\pi} d\psi P^{\rm cav}(\psi) e^{\hat{e}\cos(\psi-\phi)+\hat{m}_{ss}\{\sin(\phi)-\sin(\psi)\}\sin(\psi-\phi)}\}^{c-2},\end{aligned} \tag{65}$$

where Z_3 is the normalization constant and equal to $Z_{\rm true}$.

4. Numerical Results

In this section, we explain our method for performing the numerical calculations and show the numerical results for the dynamical behaviour. We integrate the evolution equations for the order parameters by the Euler method. We explain the method for 3OPS only for simplicity because the method for 4OPS is similar.

We specify initial conditions for m_c and e assuming $m_s = 0$ without loss of generality (as this is just a rotational symmetry of the system). Before performing the integration of ODEs, we tried various initial distributions for $P^{\rm cav}(\phi)$ and used the nonlinear Newtonian method (Brent method)[15] to solve the self-consistent equations (39-43) to determine $\hat{m}_c, \hat{m}_s, \hat{e}$

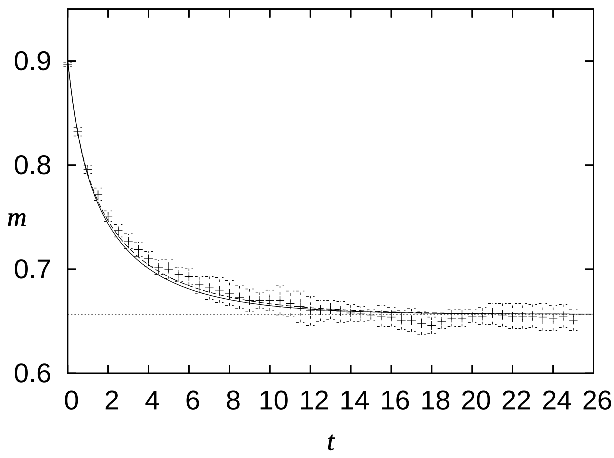

Fig. 3. Time series of the order parameter m. $c = 3$, $T = 0.7$. Solid curve: 3OPS, dashed curve: 4OPS, symbols with errorbars: simulation of the langevin eq. $N = 6,000$, average of 10 samples. Dotted line indicates the equilibrium value of m. The difference between results by 3OPS and those by 4OPS is difficult to observe because both results are within errorbars of simulation results.

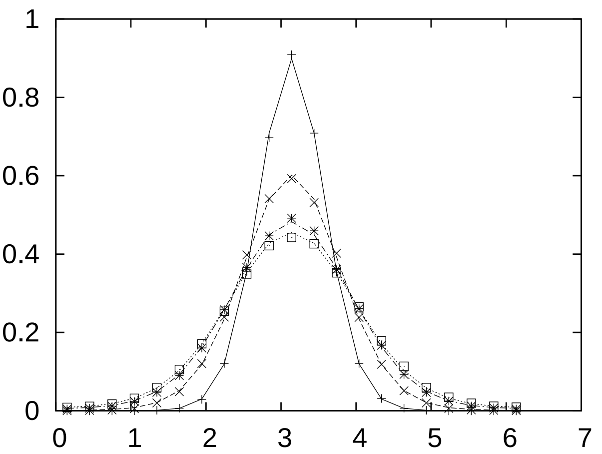

Fig. 4. Time dependence of phase distribution $P^{\mathrm{true}}(\phi)$. $c = 3$, $T = 0.7$. Curves are theoretical results (3OPS) and symbols are simulation results of the average of 10 samples for $N = 6,000$. solid curve and $+$: $t = 0$, dashed curve and $\times$: $t = 1$, dotted-dashed curve and * : $t = 5$, dotted curve and square : $t = 20$.

and $P^{\mathrm{cav}}(\phi)$. We found that $\hat{m}_s$ is nearly equal to zero, and $P^{\mathrm{cav}}(\phi)$ converges to a unimodal function and when it is redefined such that it attains maximum at $\phi = \pi$, it is almost symmetric with respect to $\phi = \pi$, as in the equilibrium case. Thus, when we began the integration of ODEs, as an initial distribution, we took a function which attains maximum at $\phi = \pi$ and satisfies $P^{\mathrm{cav}}(\pi + \phi) = P^{\mathrm{cav}}(\pi - \phi)$. Then, we estimated the time derivatives of our order parameters and estimated the value of order parameters at $t = \Delta t$ by the Euler method. This procedure is repeated for each value of $t = n\Delta t$ without assuming $P^{\mathrm{cav}}(\pi + \phi) = P^{\mathrm{cav}}(\pi - \phi)$.

As a result, we obtained distributions $P^{\mathrm{cav}}(\phi)$ and $P^{\mathrm{true}}(\phi)$ which are almost symmetric about $\phi = \pi$ (again a symmetric solution for this pair is always a solution), with $m_s \simeq 0$ and $m \simeq |m_c|$.

Next, we explain the method we used to integrate the Langevin equation. In order to compare the simulations with the theoretical results initial values for ϕ_i were generated according to the probability $P^{\mathrm{true}}(\phi)$ at $t = 0$ which is obtained by 3OPS. We took the time increment $\Delta t = 0.01$ and took the average over 10 samples.

We display the time dependence of m in Fig.3. As can be seen from the figure, the agreement between theory and simulations for $N = 6,000$ is fairly good. There is a slight difference between the result for 3OPS and that for 4OPS, but it is difficult to observe because both results are within the errorbars of the simulation result. In fig. 4, we compare the time dependence of $P^{\mathrm{true}}(\phi)$ obtained by 3OPS with that obtained by simulations for $N = 6,000$. We note that the agreement is excellent.

5. Conclusion

In this paper, we have investigated the statics and dynamics of phase oscillators on the Bethe lattice by using the cavity method.

In equilibrium, we derived a functional equation which the cavity field distribution obeys, and obtained the phase boundary curve between ordered and disordered phases in the parameter space of temperature T and the connectivity c following.[4,7] Further, we investigated the temperature dependence of the order parameter m, and found that the theoretical results agree with results of numerical simulations.

Further, we have investigated the dynamical behaviour of this system.We derived evolution equations for a 3 order parameter scheme (3OPS) and those for a 4 order parameter scheme (4OPS) and we expressed those equations in terms of the cavity field.

We found that although the time dependence of the order parameter m using 3OPS is slightly different from that for 4OPS, the difference is within the standard deviation of the result of the numerical simulation. Further, we studied the time dependence of the marginal distribution density and found that the theoretical result provided by 3OPS agrees with the result of the numerical simulation quite well.

The method and results in the present study are expected to give some useful information and suggest avenues of investigation for the study of general sparse random networks of active elements. For example, in the network of phase oscillators where each oscillator interacts with finite number of oscillators on average, the statics have been analyzed by the replica method and it has been found that when the connectivity c becomes larger, the system behaves more similarly to the present model.[12,13] The dynamical behaviour of this model can be studied by the cavity method, at least numerically, and it is now under investigation. Another example is the study of the entrainment of oscillators in the present model in which natural frequencies have a non-trivial distribution. This is a future problem.

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Appendix: Proof that the Equilibrium Solution is a Stationary States of the ODEs.

First of all, we can put $m_s = 0$ without loss of generality by a shift of the phase ϕ, $\phi' = \phi+$ constant. We use the same notation ϕ as before instead of ϕ'. The self-consistent equation for $P^{\rm cav}(\phi)$ is given by

$$P^{\rm cav}(\phi) \quad = \quad \frac{1}{Z_{\rm cav}}[\int_0^{2\pi} d\psi e^{\beta\cos(\psi-\phi)}P^{\rm cav}(\psi)]^{c-1}. \tag{A·1}$$

The marginal distribution is given by

$$P^{\rm true}(\phi) \quad = \quad \frac{1}{Z_{\rm true}}[\int_0^{2\pi} d\psi e^{\beta\cos(\psi-\phi)}P^{\rm cav}(\psi)]^{c}. \tag{A·2}$$

Since $P^{\rm cav}(\phi)$ is periodic with respect to ϕ with the interval 2π, the interval of an integration can be any interval if it's length is 2π. So, let us adopt the interval $[-\pi,\pi]$, hereafter. We assume that $P^{\rm cav}(\phi)$ *is an even function of* $\phi, P^{\rm cav}(-\phi) = P^{\rm cav}(\phi)$ which is confirmed numerically. From eq.(A·1), we obtain

$$\begin{aligned} P^{\rm cav}(-\phi) &= \frac{1}{Z_{\rm cav}}[\int_{-\pi}^{\pi} d\psi e^{\beta\cos(\psi+\phi)}P^{\rm cav}(\psi)]^{c-1} = \frac{1}{Z_{\rm cav}}[\int_{-\pi}^{\pi} d\psi e^{\beta\cos(-\psi+\phi)}P^{\rm cav}(-\psi)]^{c-1} \\ &= \frac{1}{Z_{\rm cav}}[\int_{-\pi}^{\pi} d\psi e^{\beta\cos(\psi-\phi)}P^{\rm cav}(\psi)]^{c-1} = P^{\rm cav}(\phi). \end{aligned} \tag{A·3}$$

That is, the assumption is compatible with the functional equation for $P^{\rm cav}(\phi)$. $P^{\rm true}(\phi)$ is rewritten as follows.

$$\begin{aligned} P^{\rm true}(\phi) &= \frac{1}{Z_{\rm true}}[\int_{-\pi}^{\pi} d\psi' e^{\beta\cos(\psi'-\phi)}P^{\rm cav}(\psi')]^{c-1}\int_{-\pi}^{\pi} d\psi e^{\beta\cos(\psi-\phi)}P^{\rm cav}(\psi) \\ &= \frac{Z_{\rm cav}}{Z_{\rm true}}P^{\rm cav}(\phi)\int_{-\pi}^{\pi} d\psi e^{\beta\cos(\psi-\phi)}P^{\rm cav}(\psi). \end{aligned} \tag{A·4}$$

Thus, it follows that $P^{\rm true}(\phi)$ is also an even function.

$$\begin{aligned} P^{\rm true}(-\phi) &= \frac{Z_{\rm cav}}{Z_{\rm true}}P^{\rm cav}(-\phi)\int_{-\pi}^{\pi} d\psi e^{\beta\cos(\psi+\phi)}P^{\rm cav}(\psi) \\ &= \frac{Z_{\rm cav}}{Z_{\rm true}}P^{\rm cav}(\phi)\int_{-\pi}^{\pi} d\psi e^{\beta\cos(-\psi+\phi)}P^{\rm cav}(-\psi) \\ &= \frac{Z_{\rm cav}}{Z_{\rm true}}P^{\rm cav}(\phi)\int_{-\pi}^{\pi} d\psi e^{\beta\cos(\psi-\phi)}P^{\rm cav}(\psi) = P^{\rm true}(\phi). \end{aligned} \tag{A·5}$$

Thus, $m_s = 0$ is automatically satisfied.

Now, we prove that the equilibrium values for the order parameters are really stationary solutions of the dynamical equations derived by 3OPS. In 3OPS, $P^{\rm cav}(\phi)$ satisfies the following equation.

$$P^{\rm cav}(\phi) \quad = \quad \frac{1}{Z_{\rm cav}} e^{\hat{m}_c \cos(\phi) + \hat{m}_s \sin(\phi)} [\int_{-\pi}^{\pi} d\psi e^{\hat{e}\cos(\psi-\phi)} P^{\rm cav}(\psi)]^{c-1}. \qquad (\text{A}\cdot 6)$$

The marginal distribution is given by

$$P^{\rm true}(\phi) \quad = \quad \frac{1}{Z_{\rm true}} e^{\hat{m}_c \cos(\phi) + \hat{m}_s \sin(\phi)} [\int_{-\pi}^{\pi} d\psi e^{\hat{e}\cos(\psi-\phi)} P^{\rm cav}(\psi)]^{c}. \qquad (\text{A}\cdot 7)$$

Here, we can also assume $m_s = 0$ without loss of generality. Further, as in the equilibrium case, we assume that $P^{\rm cav}$ *is an even function and* $\hat{m}_s = 0$. From the eq.(A·6), we can prove that the assumption is compatible with the functional equation. We can prove that $P^{\rm true}(\phi)$ is $P^{\rm true}(\phi)$ is also an even function. Therefore, $m_s = 0$ is automatically satisfied. The dynamical equations are given by

$$\frac{d}{dt} m_c(t) \quad = \quad -c \frac{\int_{-\pi}^{\pi} d\phi d\psi P^{\rm cav}(\phi) P^{\rm cav}(\psi) e^{\hat{e}\cos(\phi-\psi)} \sin(\phi-\psi)\sin(\psi)}{\int_{-\pi}^{\pi} d\phi d\psi P^{\rm cav}(\phi) P^{\rm cav}(\psi) e^{\hat{e}\cos(\phi-\psi)}} - T m_c(t), \qquad (\text{A}\cdot 8)$$

$$\frac{d}{dt} m_s(t) \quad = \quad c \frac{\int_{-\pi}^{\pi} d\phi d\psi P^{\rm cav}(\phi) P^{\rm cav}(\psi) e^{\hat{e}\cos(\phi-\psi)} \sin(\phi-\psi)\cos(\psi)}{\int_{-\pi}^{\pi} d\phi d\psi P^{\rm cav}(\phi) P^{\rm cav}(\psi) e^{\hat{e}\cos(\phi-\psi)}} - T m_s(t), \qquad (\text{A}\cdot 9)$$

$$\frac{d}{dt} e(t) \quad = \quad \frac{\int_{-\pi}^{\pi} d\phi \Pi_{i=1}^{c} [d\psi_i P^{\rm cav}(\psi_i)] e^{\hat{e}\sum_i \cos(\phi-\psi_i) + \hat{m}_c\cos(\phi)} (\sum_i \sin(\psi_i - \phi))^2}{\int_{-\pi}^{\pi} d\phi \Pi_{i=1}^{c} [d\psi_i P^{\rm cav}(\psi_i)] e^{\hat{e}\sum_i \cos(\phi-\psi_i) + \hat{m}_c\cos(\phi)}}$$
$$-2Te(t). \qquad (\text{A}\cdot 10)$$

At the equilibrium, $\hat{m}_c = \hat{m}_s = 0$ *and* $\hat{e} = \beta$ *should hold.* Thus, further we assume $\hat{m}_c = 0$ and $\hat{e} = \beta$. First of all, the denominator of the first terms of equations for m_c and m_s is $\frac{Z_{\rm true}}{Z_{\rm cav}}$ by the eq.(A·4)

$$\int_{-\pi}^{\pi} d\phi d\psi P^{\rm cav}(\phi) P^{\rm cav}(\psi) e^{\beta\cos(\phi-\psi)} = \frac{Z_{\rm true}}{Z_{\rm cav}} \int_{-\pi}^{\pi} d\phi P^{\rm true}(\phi) = \frac{Z_{\rm true}}{Z_{\rm cav}}. \qquad (\text{A}\cdot 11)$$

By differentiating the both sides of eq.(A·2) with respect to ϕ, we obtain

$$\frac{d}{d\phi} P^{\rm true}(\phi) \quad = \quad \frac{1}{Z_{\rm true}} c\beta [\int_{-\pi}^{\pi} d\psi' e^{\beta\cos(\psi'-\phi)} P^{\rm cav}(\psi')]^{c-1} \int_{-\pi}^{\pi} d\psi e^{\beta\cos(\psi-\phi)} \sin(\psi-\phi) P^{\rm cav}(\psi),$$
$$= \quad \beta \frac{Z_{\rm cav}}{Z_{\rm true}} c P^{\rm cav}(\phi) \int_{-\pi}^{\pi} d\psi e^{\beta\cos(\phi-\psi)} \sin(\phi-\psi) P^{\rm cav}(\psi). \qquad (\text{A}\cdot 12)$$

By exchanging variables ϕ and ψ, we obtain

$$[\beta \frac{Z_{\rm cav}}{Z_{\rm true}}]^{-1} \frac{d}{d\psi} P^{\rm true}(\psi) \quad = \quad c P^{\rm cav}(\psi) \int_{-\pi}^{\pi} d\phi e^{\beta\cos(\phi-\psi)} \sin(\phi-\psi) P^{\rm cav}(\phi). \qquad (\text{A}\cdot 13)$$

Thus, the first term of the equation for m_c is

$$-[\beta \frac{Z_{\rm cav}}{Z_{\rm true}}]^{-1} \int_{-\pi}^{\pi} d\psi \sin(\psi) \frac{d}{d\psi} P^{\rm true}(\psi) \times [\frac{Z_{\rm true}}{Z_{\rm cav}}]^{-1} = -\beta^{-1} \int_{-\pi}^{\pi} d\psi \sin(\psi) \frac{d}{d\psi} P^{\rm true}(\psi)$$
$$= -\beta^{-1} [\sin(\psi) P^{\rm true}(\psi)]_{-\pi}^{\pi} + \beta^{-1} \int_{-\pi}^{\pi} d\psi \cos(\psi) P^{\rm true}(\psi) = T m_c. \qquad (\text{A}\cdot 14)$$

Thus, it cancels with the second term of the equation, and we get $\frac{d}{dt} m_c = 0$. Next, let us consider the equation for m_s. In this case, in the numerator of the first term of the equation for $\frac{d}{dt}m_s$, by transforming variables $\phi \to -\phi$, and $\psi \to -\psi$, we note this term is 0 since $P^{\rm cav}(\phi)$ is assumed to be an even function. Since $m_s = 0$, the right-hand side of eq.(A·9) is zero. That is, $\frac{d}{dt}m_s = 0$. Now, let us consider the equation for e. The denominator of the first term of the equation for $\frac{d}{dt}e$, I_d, becomes

$$\begin{aligned} I_d &= \int_{-\pi}^{\pi} d\phi \Pi_{i=1}^{c}[\int_{-\pi}^{\pi} d\psi_i P^{\rm cav}(\psi_i)] e^{\beta \sum_i \cos(\phi-\psi_i)} = \int_{-\pi}^{\pi} d\phi \Pi_{i=1}^{c}[\int_{-\pi}^{\pi} d\psi_i P^{\rm cav}(\psi_i) e^{\beta \cos(\phi-\psi_i)}] \\ &= \int_{-\pi}^{\pi} d\phi [\int_{-\pi}^{\pi} d\psi P^{\rm cav}(\psi) e^{\beta\cos(\phi-\psi)}]^c = \int_{-\pi}^{\pi} d\phi Z_{\rm true} P^{\rm true}(\psi) = Z_{\rm true}. \end{aligned} \tag{A·15}$$

The numerator of the first term of the equation, I_n, becomes

$$\begin{aligned} I_n &= \int_{-\pi}^{\pi} d\phi \Pi_{i=1}^{c}[d\psi_i P^{\rm cav}(\psi_i)] e^{\beta \sum_i \cos(\phi-\psi_i)} (\sum_i \sin(\psi_i - \phi))^2 \\ &= \int_{-\pi}^{\pi} d\phi \Pi_{i=1}^{c}[d\psi_i P^{\rm cav}(\psi_i)] \{\frac{\partial}{\partial \phi} e^{\beta \sum_i \cos(\phi-\psi_i)}\} \beta^{-1} \sum_i \sin(\psi_i - \phi) \\ &= - \int_{-\pi}^{\pi} d\phi \Pi_{i=1}^{c}[d\psi_i P^{\rm cav}(\psi_i)] e^{\beta \sum_i \cos(\phi-\psi_i)} \beta^{-1} \{-\sum_i \cos(\psi_i - \phi)\} \\ &= T\int_{-\pi}^{\pi} d\phi \Pi_{i=1}^{c}[d\psi_i P^{\rm cav}(\psi_i)] e^{\beta \sum_i \cos(\phi-\psi_i)} \sum_i \cos(\psi_i - \phi). \end{aligned} \tag{A·16}$$

Now, let us calculate e at the equilibrium.

$$\begin{aligned} \langle e \rangle &= \langle \frac{1}{N}\sum_{i<j} c_{ij}\cos(\phi_i - \phi_j)\rangle = \frac{1}{2N}\langle \sum_{i=1}^{N}\sum_{j\neq i} c_{ij}\cos(\phi_i-\phi_j)\rangle \\ &= \frac{1}{2N} N \frac{\int_{-\pi}^{\pi} d\phi \Pi_{j=j_1}^{j_c}[\int_{-\pi}^{\pi} d\phi_j P^{\rm cav}(\phi_j) e^{\beta\cos(\phi-\phi_j)}] \sum_{j=j_1}^{j_c}\cos(\phi-\phi_j)}{\int_{-\pi}^{\pi} d\phi \Pi_{j=j_1}^{j_c}[\int_{-\pi}^{\pi} d\phi_j P^{\rm cav}(\phi_j) e^{\beta\cos(\phi-\phi_j)}]} \\ &= \frac{1}{2}\frac{\int_{-\pi}^{\pi} d\phi \prod_{j=j_1}^{j_c}[\int_{-\pi}^{\pi} d\phi_j P^{\rm cav}(\phi_j) e^{\beta\cos(\phi-\phi_j)}] \sum_{j=j_1}^{j_c}\cos(\phi-\phi_j)}{Z_{\rm true}}. \end{aligned} \tag{A·17}$$

Thus, $\frac{I_n}{I_d} = 2T\langle e \rangle$. Therefore, we obtain $\frac{d}{dt}e = 0$. Finally, $\langle e \rangle$ can be further rewritten as

$$\begin{aligned} \langle e \rangle &= \frac{c}{2} Z_{\rm true}^{-1} \int_{-\pi}^{\pi} d\phi [\int_{-\pi}^{\pi} \Pi_{j=j_1}^{j_c} d\phi_j P^{\rm cav}(\phi_j) e^{\beta\cos(\phi-\phi_j)}] \cos(\phi - \phi_{j_1}) \\ &= \frac{c}{2} Z_{\rm true}^{-1} \int_{-\pi}^{\pi} d\phi \int_{-\pi}^{\pi} d\phi_{j_1} P^{\rm cav}(\phi_{j_1}) e^{\beta\cos(\phi-\phi_{j_1})} \cos(\phi-\phi_{j_1}) [\int_{-\pi}^{\pi} d\psi P^{\rm cav}(\psi) e^{\beta\cos(\phi-\psi)}]^{c-1} \\ &= \frac{c}{2} Z_{\rm true}^{-1} \int_{-\pi}^{\pi} d\phi \int_{-\pi}^{\pi} d\phi_{j_1} P^{\rm cav}(\phi_{j_1}) e^{\beta\cos(\phi-\phi_{j_1})} \cos(\phi-\phi_{j_1}) Z_{\rm cav} P^{\rm cav}(\phi) \\ &= \frac{c}{2}\frac{Z_{\rm cav}}{Z_{\rm true}} \int_{-\pi}^{\pi} d\phi \int_{-\pi}^{\pi} d\psi P^{\rm cav}(\phi) P^{\rm cav}(\psi) e^{\beta\cos(\phi-\psi)} \cos(\phi-\psi) \\ &= \frac{c}{2}\frac{\int_{-\pi}^{\pi} d\phi \int_{-\pi}^{\pi} d\psi P^{\rm cav}(\phi) P^{\rm cav}(\psi) e^{\beta\cos(\phi-\psi)}\cos(\phi-\psi)}{\int_{-\pi}^{\pi} d\phi d\psi P^{\rm cav}(\phi) P^{\rm cav}(\psi) e^{\beta\cos(\phi-\psi)}} = \frac{c}{2}\langle \cos(\phi-\phi')\rangle_2^{\rm 3OPS}. \end{aligned} \tag{A·18}$$

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